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# Typical Block Diagrams for a Transistor Digital Computer

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**Synopsis:** The first electric digital computers were built around the properties of relays. The superior speed capabilities of vacuum tubes has led in recent years to their use in new computer designs to replace relays. Because of the small size, low power consumption, and expected long life of transistors, it now appears that the transistor will replace the vacuum tube as a computer element. This paper presents a study of binary computer functions with recommended mechanizations that were selected because they appeared to be readily attainable with transistors now under development. Block diagrams are presented of switches, memory units, arithmetic units, and other basic components. Estimates are given for the number of parts required in the units. It is concluded that a high-performance all-semiconductor computer can be built with germanium diodes and transistors.

**D**IGITAL computer operations<sup>1</sup> can be divided into two classes, memory and logic. Memory includes the three operations write, read, and erase, where the write operation, for example, can be defined as the representation in space of a function of time. In computers, the logic operations consist of the recognition of spatial distributions of voltages and currents. All the logic operations that can be described in words can be mechanized with suitable combinations of three basic functions. The basic functions are "or," "and," and "inhibition" or negation. An  $n$ -terminal or-circuit will have an output when any one of its  $n$  input leads is energized. An  $n$ -terminal and-circuit will have an output only if all of its  $n$  leads are energized. Inhibition leads can be added to either of the two circuits with the result that the circuit cannot have an output as long as there is a signal on any of the inhibition leads. These logic circuits have the advantage that they can be built out of passive nonlinearities. A nonlinear device has two states, a high-impedance and a low-impedance state. The more pro-

nounced the nonlinearity the more satisfactory the unit is as a 2-state element. Since nonlinearity can be obtained from purely passive elements, it is possible to obtain all of the logic functions of a computer without the use of vacuum tubes or other active elements. In practice, it is necessary to use some active elements because the nonlinear devices available are not ideal and have losses. It is necessary, therefore, to use amplifiers to make up for losses in the logic circuits. It is important to note that as the efficiency of the logic circuits is increased, the number of amplifiers required is decreased.

This computer design philosophy was followed in the design of the National Bureau of Standards Computer SEAC.<sup>2</sup> It is believed that the approach that will result in a vacuum-tubeless computer at the earliest date is to follow the SEAC example, in so far as the use of germanium diode logic circuits is concerned, but replacing the vacuum-tube amplifiers with transistor amplifiers. Since the transistor itself has voltage and current relationships quite similar to a germanium diode, it is expected that the diodes in a transistor computer will operate in a more natural environment than in a vacuum-tube computer and will respond to such favorable conditions by exhibiting longer life and more reliable operation. It is for such an all-semiconductor computer that the block diagrams described herein have been planned.

Serial operation has been used in all of these designs. In serial operation all the digits of a word or number are transmitted in series along a single wire with the least significant digit first. The interval between the rise of successive digit pulses is referred to as one digit time. The interval between successive numbers or words is commonly referred to as the word time. Digit times of 1 microsecond have proved to be practicable, and words as long as 50 binary digits have been employed (equivalent to 15 decimal digits).

A number of the block diagrams presented herein have been mechanized with a transistor amplifier that regenerates pulses at a megacycle rate. This amplifier operates on a total power drain of approximately 50 milliwatts. The details of

this amplifier and the associated logic circuits have been presented by the writer in another paper.<sup>3</sup>

The reasons for choosing serial instead of parallel operation are quite simple. In a parallel system, each digit of a number appears in a different circuit. In a parallel adder a separate adder stage is used for each digit, whereas in a serial adder only one adder stage is used, and this one stage handles all the digits. Not only are the arithmetic operations performed with fewer components in the serial machine but switching is also simplified. To switch an  $n$ -digit parallel number requires  $n$  switches to switch the separate wires, whereas in the serial machine only one switch is required.

## Basic Building Blocks

The proposed computer elements are based on an assembly of a relatively small number of different kinds of basic units. The basic units have been limited to an or-circuit, an and-circuit, an inhibitor-circuit, an amplifier, and a storage cell based on a delay line. The designs suggested are similar to those in the EDVAC designed at the University of Pennsylvania and the Bureau of Standards SEAC. Some of the basic components have been described in the literature.<sup>4,5</sup> They are described again in the following paragraphs to provide a specific basis for the estimates of numbers of parts that are a goal of this study.

An  $n$ -terminal or-circuit (see Figure 1) develops an output when any one of its input terminals is energized. Crystal diodes are put in series with each input to prevent a pulse at one input from feeding back to any of the other inputs. An  $n$ -terminal circuit, therefore, requires  $n$  crystal diodes.

An and-circuit having  $n$  input ter-

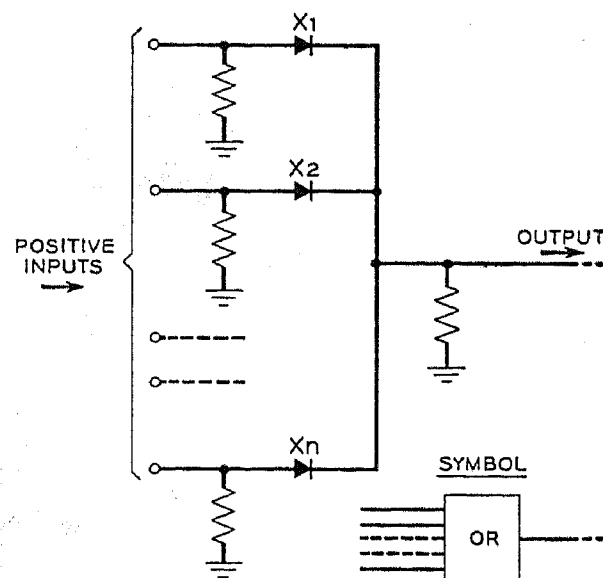


Figure 1. Or-circuit

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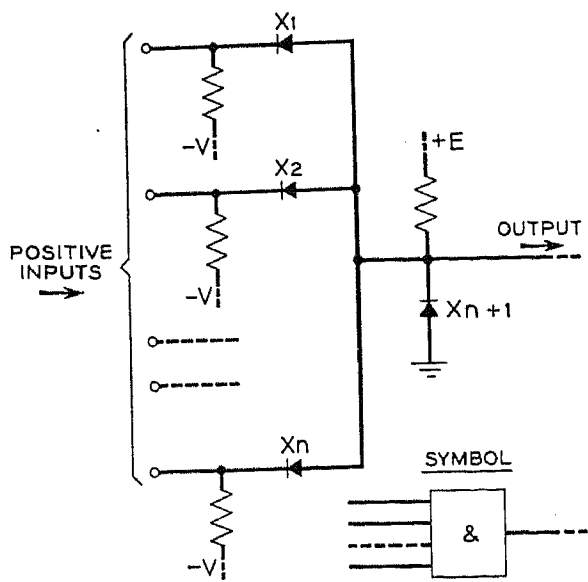


Figure 2. And-circuit

minals develops an output only when all  $n$  of the input terminals are energized. In Figure 2 each of the inputs is returned to a negative voltage and the output is clamped slightly below ground by  $X_{n+1}$ . Only when all of the inputs rise above ground will  $X_{n+1}$  be cut off, permitting the output to rise. Thus the output consists of the overlapping part of the inputs. An  $n$ -terminal circuit is seen to require  $n+1$  crystals.

Two terminal and-circuits are used extensively in serial computers for retiming signals. One terminal of the circuit is fed by the signal to be retimed while the other is fed by digit pulses from a master clock. There will be an output from the circuit only for the overlap of the master digit pulse and input signal. The circuit whereby the output is made a replica of the pulse from the clock is discussed in connection with the basic amplifier.

An inhibitor terminal can be added to any and-circuit or or-circuit. Such a circuit operates as though there were no inhibitor terminal when the inhibiting pulse is absent. When the inhibiting pulse is present, however, the circuit prevents any output from being developed. The inhibiting circuits used in the designs

proposed herein are of the simple variety shown in Figure 3 where positive inputs synchronized in time are required. Note that the signal to be inhibited is passed through an eighth-digit delay line, while the inhibiting pulse is passed both through and around a quarter-digit delay line. This insures that the inhibitor pulse will, in effect, arrive earlier than the signal pulse and last longer. In the absence of input pulses, crystal diode  $X_4$  will clamp the output at ground because input  $B$  is returned to a negative potential. Note that  $X_1$  and  $X_2$  are returned through the transformer to a positive potential. If input  $B$  goes positive (without an inhibiting pulse appearing on input  $A$ )  $X_4$  will be cut off and the output voltage will rise until it is clamped at the positive potential to which  $X_1$  and  $X_2$  are returned. If there is an inhibiting pulse (positive) it is inverted by the transformer and will carry  $X_1$  and  $X_2$  negative, which will keep  $X_4$  conducting, no matter what happens at  $B$ . Thus, if pulses  $A$  and  $B$  were written as a 2-digit binary number  $AB$ , the circuit translates 01 into a 1 at the output. It translates 00, 10, and 11 into zero at the output.

Active elements will be used not as flip-flops or switches but as repeating amplifiers to make up for attenuation in crystal diode circuits and delay lines. The standard use will include a retiming feature as well as amplification. Wherever a pulse is likely to suffer intolerable attenuation, deformation, or a variable delay, a circuit like that of Figure 4 is inserted in the machine.

The assembly shown has two inputs,  $A$  and  $B$ . Input  $A$  is the pulse to be retimed and amplified. Input  $B$  comes from the master clock. This component supplies reference pulses (known as digit pulses) every digit time. These pulses are available in various phases, that is, with various but accurately controlled delays of a fraction of a digit time. The pulse fed to  $B$  is selected to rise sometime between the expected rises and falls of the pulses on  $A$ .

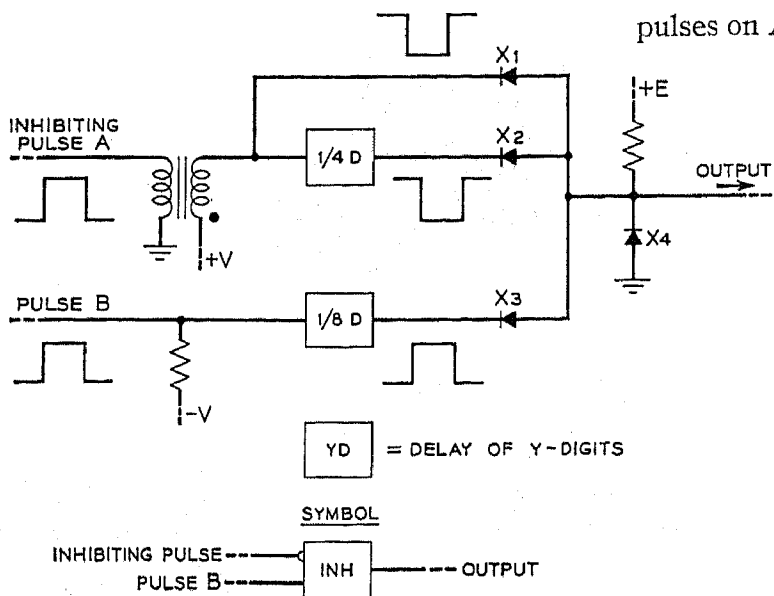


Figure 3 (left). Inhibitor-circuit

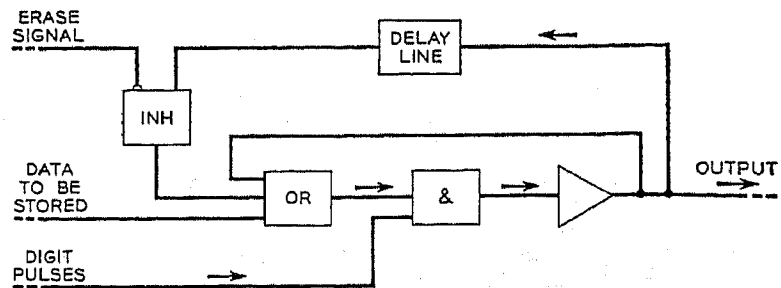


Figure 5 (right). Storage cell

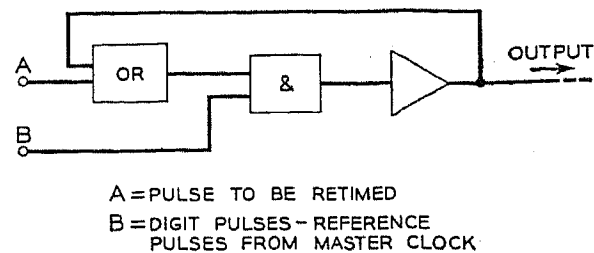


Figure 4. Amplifier with pulse retiming

If there is no input on  $A$ , there will be no output from the amplifier because of the and-circuit. If there is an input on  $A$ , when the digit pulse arrives the amplifier output will rise with a rise time determined by the digit pulse (assuming that the amplifier passband does not limit it). Part of the amplifier output is fed back through an or-circuit to the and-circuit. This insures that the output pulse will not fall until the reference digit pulse does, even though pulse  $A$  may have ended after  $B$  rose.

This reshaping with logic circuits and an amplifier is the way in which every pulse is maintained with the desired time synchronization. Pulse  $A$  may vary somewhat in the delay it has suffered but the output pulse still will leave the amplifier at a time determined only by the reference pulse from the master clock. Thus, the pulses in the computer are made to have fixed durations and to occur at designated times.

In the tabulations of components in the following paragraphs an active element is intended to represent as one unit a vacuum tube with transformer or a transistor. The term will include neither the and-circuit nor the or-circuit shown in Figure 4.

The basic storage cell proposed is not a static device like a flip-flop but is an electric delay line plus an amplifier. When vacuum tubes are used, this type of storage saves one active element in a 1-digit storage cell and is believed to be a more reliable use of active elements. In larger storage units more elements will be saved.

A block diagram of a cell is shown in Figure 5. The unit has three inputs: digit pulses, the signal to be stored, and an erase pulse. The digit pulses are received from the master clock every digit time and are used to retime the output of the

Table I. Basic Building Blocks

Unit	Crystal Diodes	Digits of Delay	Active Elements
N-terminal or -circuit.....N.....	0	0	0
N-terminal and -circuit..N+1.....	0	0	0
Inhibitor-circuit.....4.....	$3/8$	0	0
N-digit storage cell.....10*.....N.....	1*		

\* One active element will be used for  $N \leq 8$ ; for  $N \geq 8$  the number of active elements planned is the smallest integer  $\geq N/8$ . Six additional crystal diodes will be required with each active element.

delay line before it is amplified and recirculated. The erase signal is received whenever new data are to be stored. It serves to erase the data in storage, blocking the delay line output from its input until the new data have been inserted.

The delay line may be long enough to store one word or just one digit of data. It is believed that up to 15-digit delay lines with lumped impedances can be built that will hold the delay constant to within a small fraction of one digit time. Depending on the length of a word, it may be necessary to break 1-word lines into sections and insert an amplifier between sections to retime and regenerate the pulses stored. To insure conservative estimates, the estimates made in following sections are on the basis of regeneration after every eight digits of delay.

The circuits which have been discussed use the components listed in Table I. This table will be used frequently in estimating the parts required in the larger assemblies.

## Switches

The switches are planned to combine a switching and a storage function. When a switch is given instructions to go to the  $k$ 'th position, it goes there and it remembers that it is to remain there (self-locking operation). All the elements of the switches have been discussed in preceding paragraphs. How these elements are combined to make switches is described in the following:

A single-pole double-throw switch, as shown in Figure 6, consists of a storage cell and a switch unit. When a 1 is stored in the storage unit, as the result of a pulse on the switching instruction lead, the left-hand and-circuit of the switch unit will pass signal  $a$  while the inhibitor blocks signal  $b$ . When a zero is stored, the and-circuit will block signal  $a$  and the inhibitor-circuit in the switch unit will pass signal  $b$ .

Whenever the switch is to be reset, an erase signal is fed to the storage unit, which then drops its old instruction and

goes to position  $b$  unless the new instruction sets it to  $a$ .

The switch unit itself uses 13 crystal diodes,  $3/8$  digit of delay, and one active element, as can be ascertained from Figure 6 and Table I. A complete single-pole double-throw switch has 23 diodes,  $1^{3/4}$  digits of delay, and two active elements. A 2-pole double-throw switch would have two switch units and one storage unit. A 3- or 4-pole switch would have three or four switch units, one storage unit, and perhaps an extra amplifier to prevent the switch units from loading down the storage unit excessively.

Switches with more than two positions can be assembled in a slightly different manner from the double-throw switches. Suppose, for discussion, that an 8-position switch is needed. A 3-digit code must be sent to the switch to specify the position it is to select. A convenient way to operate such a switch is to translate the 3-digit code into a 6-wire code so that each digit is represented by signals of opposite phase on a pair of wires. In this system, a 1 is represented by a positive pulse on the positive bus and a negative pulse on the negative bus. A zero is represented by a negative pulse on the positive bus and a positive pulse on the negative bus. A 3-terminal and-circuit is provided for each of the eight lines that may be selected. The and-circuits are operated by positive pulses, and all are connected to either one or the other of each pair of wires representing the three digits of the position code. The and-circuit on the fifth (101) wire, for example, is connected to the positive bus of the pair representing the coefficient of  $2^2$ , to the negative bus of  $2^1$ , and the positive bus of  $2^0$ . The three leads of the and-circuit will be energized by positive pulses only when 101 is sent to the switch as instructions.

A single-pole 8-position switch is shown in Figure 7. The eight 3-terminal and-circuits (one for each horizontal lead), plus the 9-terminal or-circuit make an easily studied crystal matrix which feeds another and-circuit for retiming of pulses and an amplifier on the output. The and-circuits are fed by switches  $S_0$ ,  $S_1$ , and  $S_2$ , which are driven by the switching instructions. These three switches are similar to the one shown in Figure 6.  $S_0$  is operated by the coefficient of  $2^0$ ,  $S_1$  by the coefficient of  $2^1$ , and  $S_2$  by the coefficient of  $2^2$ . As the figure is drawn, only the and-circuit on line 7 (111) is energized. A pulse will be received at the output whenever a pulse is put on line 7. Pulses can be put on any of the other lines without getting to the output, because every other line is held negative by at least one lead

Table II. Switches

Type of Switch	Crystal Diodes	Digits of Delay	Active Elements
1P.2T .....	23	$1^{3/4}$	2
2P.2T .....	36	$1^{3/4}$	3
4P.2T .....	63	$6^{1/2}$	6
1P.4T .....	61	$2^{3/4}$	4
1P.8T .....	105	6	7
1P.16T.....	160	7	9
1P.32T.....	311	$8^{3/4}$	11

Where 2P.2T, for example, represents a 2-pole 2-throw switch.

from  $S_0$ ,  $S_1$ , or  $S_2$ . If an erase signal were sent to the three storage cells of  $S_0$ ,  $S_1$ , and  $S_2$ , the number 111 would be erased, leaving 000, and the switches would be free to move to new positions and select another one of the eight inputs.

Based on this type of mechanization, an  $r$ -position switch requires  $n$  single-pole double-throw switches, where  $n$  is the smallest integer equal to or greater than  $\log_2 r$ . The switch requires  $r(n+1)$  crystal diodes, plus four diodes for retiming the output, and an output amplifier.

Table II gives estimates of the crystal diodes, digits of delay, and active elements required in switches of different degrees of complexity.

## Handling of Negative Numbers

The most significant digit place of every number is reserved to indicate the sign of the number. Positive numbers have a zero in the last place. A negative number is obtained by taking the 2's complement of the positive number. This results in every negative number having a 1 in its last place. The system is equivalent to the 10's complement method used in decimal calculators. In a decimal calculator operating with three significant figures, a fourth place might be provided for the sign. The number  $-187$  might be represented by its 10's complement 9,813. Then, for example, if  $-187$  were required to be added to 500 the operation would be to add 9,813 to 500, which gives 10,313, and thus is recognized as 0,313 since the machine was assumed to have only four digit places.

Table III. Binary Addition

Addend	Inputs		Outputs	
	Augend	Carry	Sum	New Carry
0.....0.....0.....0.....0				
0.....0.....1.....0.....0				
0.....1.....0.....0.....0				
1.....0.....1.....0.....0				
0.....1.....1.....0.....0				
1.....1.....0.....0.....0				
1.....0.....1.....0.....0				
1.....1.....1.....1.....1				

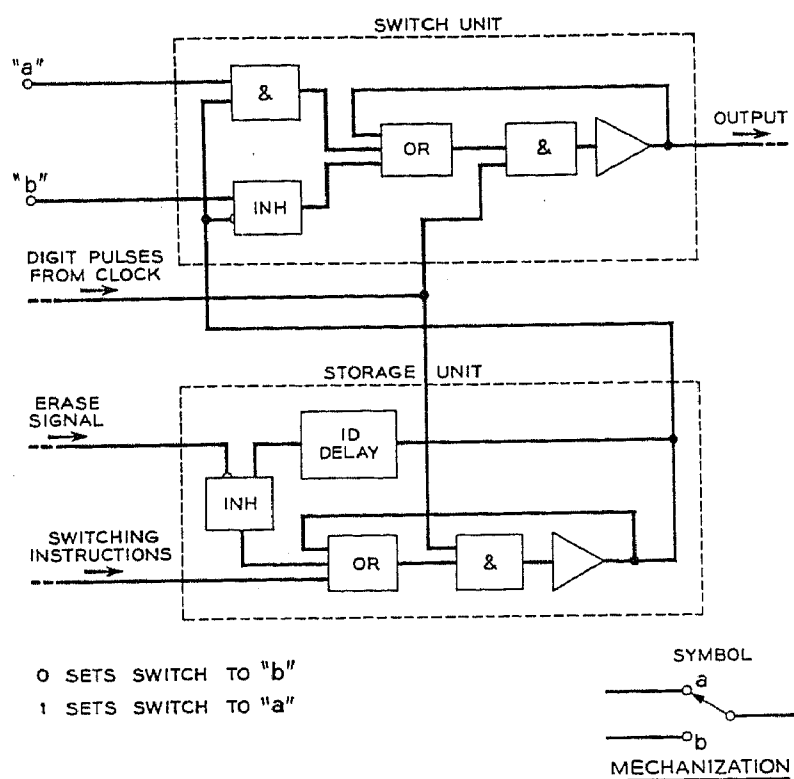


Figure 6 (left).  
A simple switch

(a) FORMATION OF NEGATIVE NUMBERS

+8 = 0001000  
ONES  
COMPLIMENT = 1110111  
ADD ONE  
∴ -8 = 1111000  
CHECK -8+8=0  
+8 = 0001000  
-8 = 1111000  
SUM = 10000000 = ZERO  
TO THE MACHINE

(b) ADDITION

-8+5=-3  
-8 = 1111000  
+5 = 0001001  
SUM = 1111101 = -3  
CHECK +3 = 0000011  
SUM = 10000000 = ZERO  
-8+15=7  
-8 = 1111000  
+15 = 0001111  
SUM = 10000111 = 7

(c) MULTIPLICATION

-8x7=-56  
-8 = 1111000  
7 = 0000111  
1111000  
0000111  
1111000  
1111000  
1111000  
0000000  
0000000  
0000000  
0000000  
0000000  
1001000  
↑ SHOWS ANSWER IS NEGATIVE

CHECK  
-56+56=0  
-56 = 1001000  
+56 = 1110000  
1001000  
1110000  
1000000

(d) ANOTHER MULTIPLICATION

-8x-7=+56  
-8 = 1111000  
-7 = 1111001  
1111000  
1111001  
0000000  
0000000  
1111000  
1111000  
1111000  
1111000  
0111000  
↑ INDICATES POSITIVE NUMBER

Figure 8 (right).  
Handling of negative numbers

NEGATIVE NUMBERS IN A 6  
SIGNIFICANT DIGIT SYSTEM  
WITH ONE IN SEVENTH PLACE  
SIGNIFYING THAT THE NUMBER  
IS NEGATIVE

A negative number (2's complement) can be obtained in the binary computer by first, forming the 1's complement (changing all zeroes to 1's and vice versa by means of an inhibitor circuit), and then adding 1. Figure 8 shows several examples of binary arithmetic performed with negative numbers.

A point of interest is that even though either or both the multiplicand and multiplier are negative the correct sign will be obtained for a product provided the negative numbers are increased from their normal length  $W$  to  $2W-1$  by filling in 1's before multiplying. This is necessary be-

cause the product of two  $W$ -digit numbers, where the last digit specifies sign, is a number  $2W-1$  digits long. Unless the multiplicand and multiplier are increased to length  $2W-1$  (when negative) the  $2W-1$  place may be incorrect. This lengthening is not required for positive numbers, because the digits from  $W$  to  $2W-1$  would be zeroes if they were filled in and would contribute nothing to the product. A consequence is that the product  $XY$  can be obtained to  $2W-1$  places with  $W(W-1)$

elementary additions provided  $X$  and  $Y$  are positive while  $W(2W-1)$  additions will be required if both are negative. Where fast multiplication is desired it may be advisable to convert negative numbers to positive ones, multiply, and then adjust the sign of the product.

## Adder

The adder can be considered as a translator with three inputs: addend, augend, and carry. It is a simple translator in that its output is a function only of the number of ones among its three inputs, as can be seen from Table III.

The combination 0 0 0 is automatically taken care of in the adder shown in Figure 9. The three dashed circuits at the left of the block diagram recognize three situations among the three inputs. The situations are: at least one 1, at least two 1's and three 1's among the inputs. If there is only one 1, it will go through the bottom or-circuit, the following inhibitor circuit, and then another or-circuit. After being reclocked and amplified it will provide a 1 as the sum. In this case none of the and-circuits on the  $A$ ,  $B$ , and carry leads will have operated. If there are at least two 1's on the  $A$ ,  $B$ , and carry leads at least one of the three 2-terminal and-circuits in the dashed box will operate with two results. The output of the 3 terminal or-circuit at the bottom left of the diagram will be inhibited so that it makes no contribution to the sum. In addition, a carry signal will be developed which is delayed one digit, reclocked, and amplified to serve as the carry for the

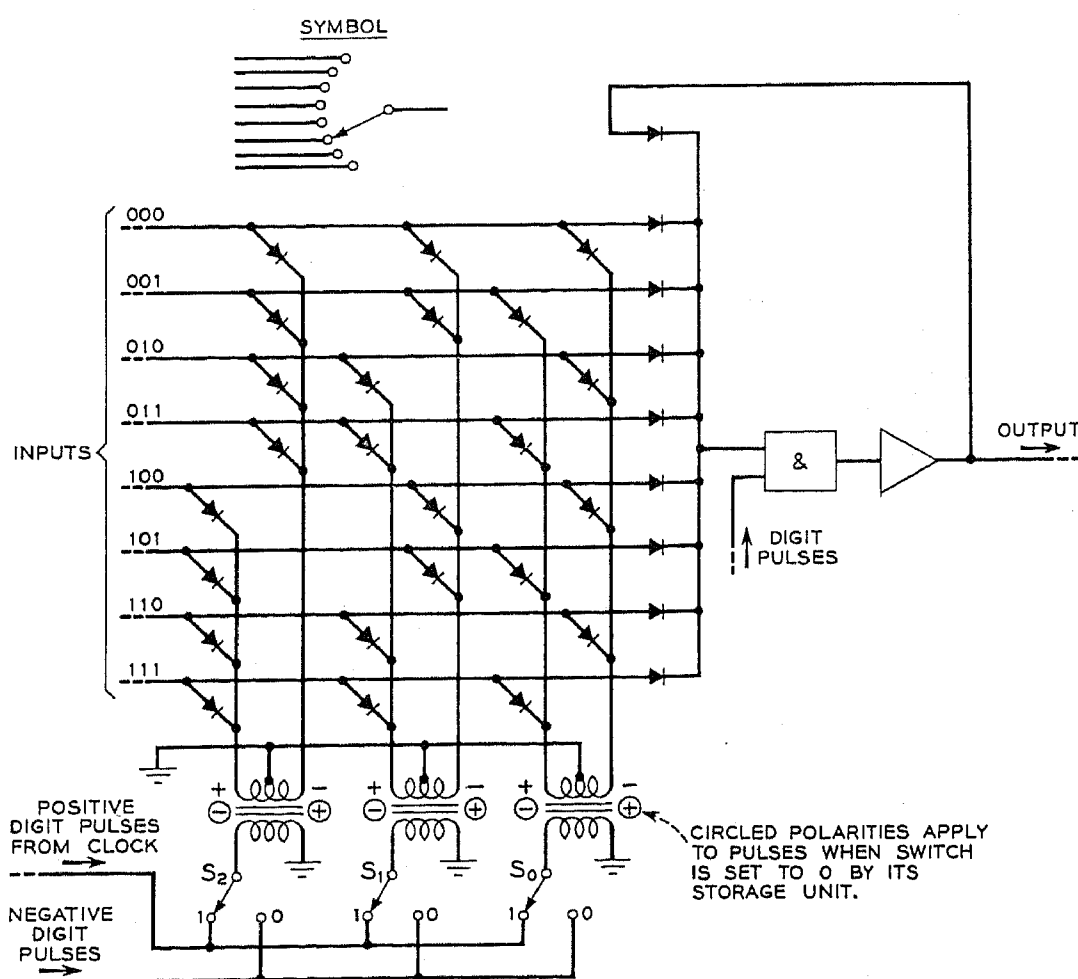


Figure 7. An 8-position switch

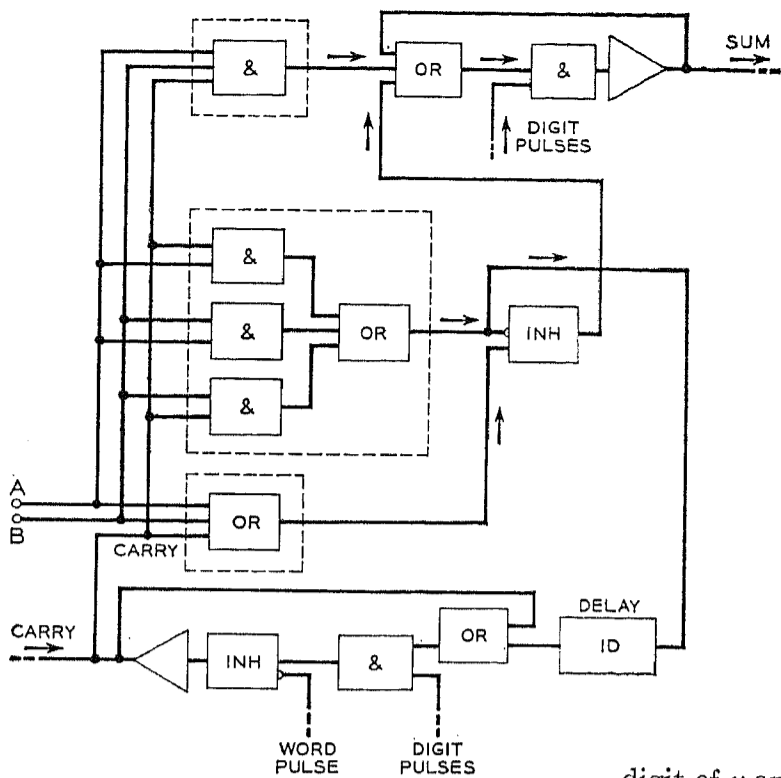
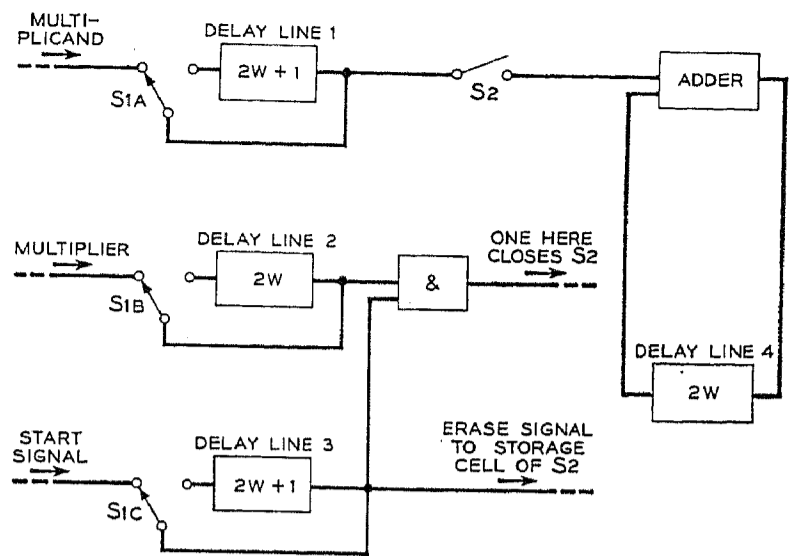


Figure 9 (left).  
Adder

Figure 11 (right).  
Multiplier



next augend and addend. If there are three 1's, the 3-terminal and-circuit at the top and left-hand side of the diagram will operate and develop a sum of one. The three 2-terminal and-circuits on the left of the diagram also will have operated and provided the carry. Thus, the adder table is mechanized with two active elements (amplifiers).

The inhibitor circuit in series with the carry lead should be noted. This circuit is fed by a word pulse as well as the carry digits. The word pulse is received in synchronism with the first digit of every number. The word pulse will inhibit the carry pulse if one is present and will prevent a carry developed in one problem from being used in the next. This feature is required in the addition of negative numbers.

The carry lead is brought outside the adder to facilitate subtraction. Suppose  $x$  is to be subtracted from  $y$ . The number  $y$  might be fed to the addend terminal and the number  $x$  fed through an inhibitor circuit to the augend terminal. The inhibitor would also be fed by digit pulses from the master clock, and the augend would therefore be the 1's complement of  $x$ . A 1 would be inserted into the carry terminal in synchronism with the first

digit of  $y$  and of the 1's complement of  $x$ . The sum out of the adder would then be  $y - x$ . The adder produces a sum within a fraction of a digit time after it receives an input. Thus, there is only a small delay in obtaining the sum of two numbers.

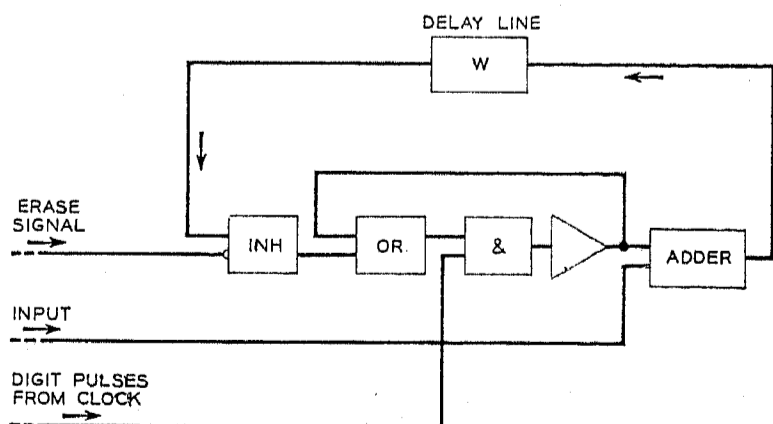
Table I can be consulted to show that the adder of Figure 9 requires 38 crystal diodes,  $1\frac{3}{8}$  digits of delay, and two active elements. The operation of so many crystal diode circuits in series without amplifiers may be questioned, but it is believed that it will become feasible as diodes and transistors are improved.

### Accumulator

A block diagram of an accumulator is shown in Figure 10. The output of the adder is fed back to its input through a  $w$ -digit delay line. The output of the delay line is reclocked continuously in the and-circuit in accord with digit pulses from the master clock. The timed pulses then are amplified in a 1-stage amplifier to make up for attenuation in the delay line. Whenever a new accumulation is to be started, an erase signal is fed to the inhibiting circuit. This signal is  $w$ -digits long and blocks the output of the delay line from the adder and insures that the new accumulation will start from zero. An accumulator to accumulate sums 48 digits

Figure 10 (left).  
Accumulator

Figure 12 (right).  
Operation of  
multiplier



long will require 76 crystals, 48 digits of delay, and eight active elements.

### Multiplier

The multiplier that is discussed in the following paragraphs is designed to multiply two positive members, each having  $W$  digits, in  $W(2W+1)$  digit times. Operating at a megacycle rate, the product of two 48-binary digit numbers would be obtained in 4,656 microseconds. It is believed most efficient to convert all negative members to positive ones before multiplying and to adjust the sign of the product according to the rules of algebraic multiplication. The components to be added to the multiplier to make the sign correction have not been included in this study. The multiplicand  $x$  is multiplied by the first (least significant) digit of the multiplier  $y$  in the first  $(2W+1)$  period and by the  $W$ 'th digit of  $y$  during the  $W$ 'th  $(2W+1)$  period. The  $W$  partial products are added together, with each partial product moved over one place with respect to the preceding one as it is accumulated.

The multiplication table in binary arithmetic is very simple. If it is desired to multiply the binary number  $x$  by a single binary digit  $m$ ,  $x$  is connected to the input of a switch that is held open if  $m=0$ , and closed if  $m=1$ . The output of the switch will be  $mx$ . Thus, in multiplying  $x$  by the successive digits of the number  $y$ , it is only required to pass  $x$  through a switch that is opened and closed at the appropriate times according to whether the successive digits of  $y$  are zeroes or 1's.

TIME	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
MULTIPLICAND	0	1	1					0	1	1				0	1	1	
MULTIPLIER	1	0	1				1	0	1				1	0	1		
START	1							1							1		
SWITCH	←--CLOSED--→						←--OPEN--→						←--CLOSED--→				
ADDEND	0	1	1					0	0	0					0	1	1
AUGEND	0	0	0				0	1	1				0	1	1	0	
SUM	0	1	1				0	1	1	0			0	1	1	1	1

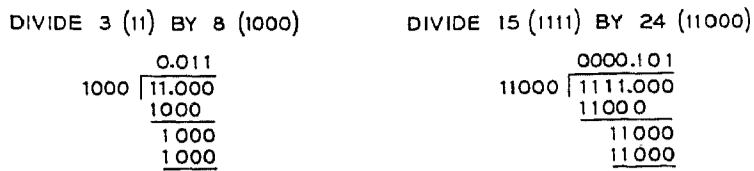


Figure 13 (above). Binary division

Figure 15 (right). Divider

The block diagram of a multiplier is shown in Figure 11. The multiplicand  $x$  is stored in delay line 1, and the multiplier  $y$  is stored in delay line 2. After the first word period,  $S1-A$ ,  $S1-B$ , and  $S1-C$  are thrown to position  $b$  and the multiplier, multiplicand, and start pulse recirculate in their delay lines until the multiplication is finished.  $DL1$  (delay line 1) storing the multiplicand has one more digit of delay than line  $DL2$  in which the multiplier is stored. Therefore, after the first circulation of  $x$  through  $DL1$ , the least significant digit of  $x$  leaves  $DL1$  just as the second least significant digit of  $y$  leaves  $DL2$ . After the  $W-1$ 'th circulation of  $x$ , its least significant digit leaves  $DL1$  just as the most significant digit of  $y$  leaves  $DL2$ .

The output of  $DL2$  is fed continuously to an and-circuit. Every  $2W+1$  digits, an examining pulse is fed to the and-circuit from  $DL3$ , and if the digit of  $y$  coming out of  $DL2$  is a 1 at that time, the and-circuit develops a 1 in its output. If, when the examining pulse comes along, the coefficient of  $y$  leaving  $DL2$  is a zero, the and-circuit will have no output. Because  $DL2$  has a delay of  $2w$  digits, the and-circuit in effect filters out the successive

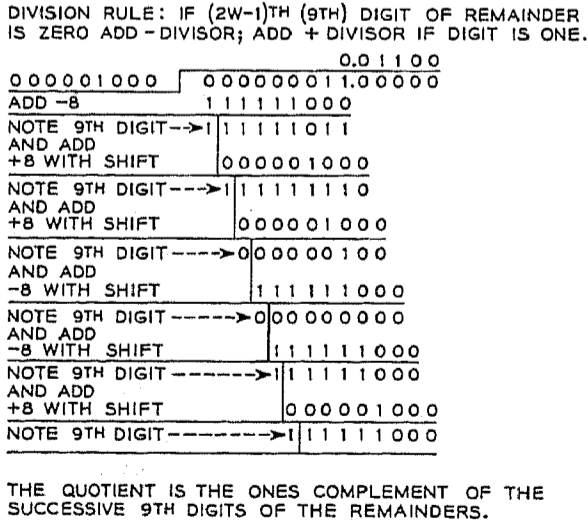
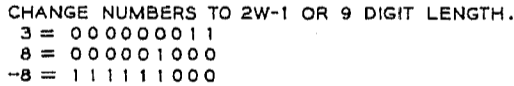
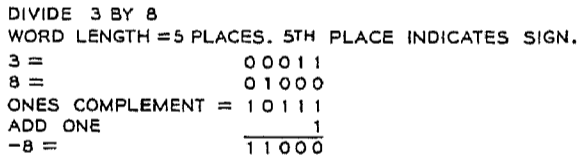
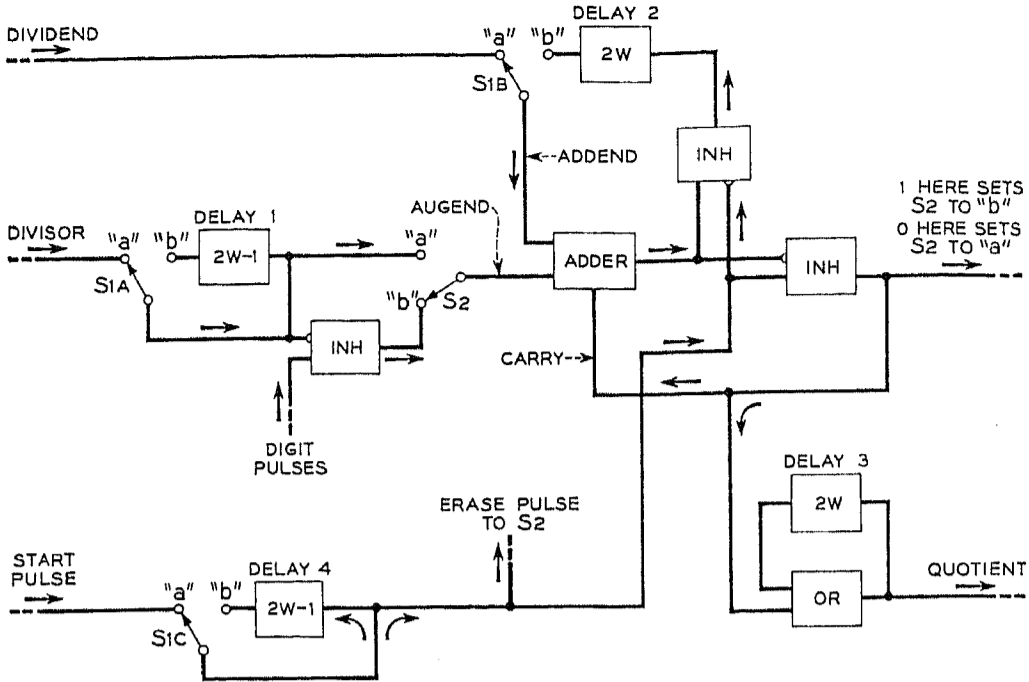


Figure 14. Simplified division process



digits of  $y$ , choosing a new digit every  $2w+1$  digit times. The examining pulse is also sent to  $S2$ , where it is the erase signal of the storage cell associated with  $S2$ . The output of the and-circuit goes to  $S2$  as switching instruction and is stored there. Thus,  $S2$  stores either a 1 or a zero, according to whether the digit of  $y$  examined is a 1 or a zero. The switch is arranged so that when a 1 is stored,  $S2$  is closed, and when a zero is stored, it is open. The examining pulse is developed from the start pulse which is received by the multiplier coincident with the first digits of  $x$  and  $y$  and recirculated in delay line 3.

A new partial product is obtained at the output of  $S2$  every  $2w+1$  digit times by the process just described. To accumulate them, an adder is provided which accepts as one input the output of  $S2$ , and as the other input its own sum output delayed by  $2w$  digits. The one digit placedifference in the arrival of the two inputs of the adder automatically provides the "shift" feature necessary in adding the partial products. Figure 12 shows a digit by digit account of the operation of a 3-digit multiplier when multiplying 110 by 101.

A multiplier based on Figure 12 can be mechanized for 48 digit words with 390 crystals, approximately 390 digits of delay, and 55 active elements.

### Divider

The divider operates with repeated subtractions and shift operations much as one would do in pencil and paper division. Figure 13 shows two division examples. In the first one, 3 is divided by 8. First, it is seen that 1,000 will not go into 11, so the first zero is written in the quotient. 100.0 will not go into 11, so the second zero is written. 10.00 will go into 11.0, so

a one is written in the quotient and 10.00 is subtracted from 11.00. The remainder is 1.00. The binary point is again shifted in the divisor, giving 1.000 which goes exactly once, and another one is written in the quotient giving an answer of 0.011 or  $0 \times \frac{1}{2} + 1 \times \frac{1}{4} + 1 \times \frac{1}{8}$  equal to  $\frac{3}{8}$ .

It should be noted that at each step the divisor is either subtracted from the remainder or nothing is subtracted and that a shift to the right is made each time before the divisor is subtracted. In effect, the successive subtrahends are obtained by repeated divisions of the divisor by 2. For machine operation, the awkward part of this process is that it cannot be ascertained whether a subtraction should be made without making it and seeing whether or not the remainder is positive or negative. If the remainder is negative, the subtrahend should be added back. This necessity for sometimes adding back the subtrahend may be avoided. Let  $x$  be the subtrahend and  $R$  the minuend. Suppose  $x$  is subtracted from  $R$  and the remainder found to be negative. During the next step the machine should subtract  $x/2$  from  $R$ , but the machine has lost  $R$  by subtracting  $x$  from it.  $R$  can be regained and  $x/2$  subtracted at the same time by adding the factor  $x/2$  to the negative remainder  $R-x$  which gives the desired result  $R-x/2$ . Thus, on the successive steps the machine either subtracts half the previous subtrahend or adds half the previous subtrahend, depending on whether the previous remainder was positive or negative. The process is simplified further by avoiding subtraction through adding the 2's complement of the subtrahend rather than subtracting the subtrahend.

An example of the simplified division process is shown in Figure 14. The machine is assumed to have five digit places. The first step is to fill out the divisor and

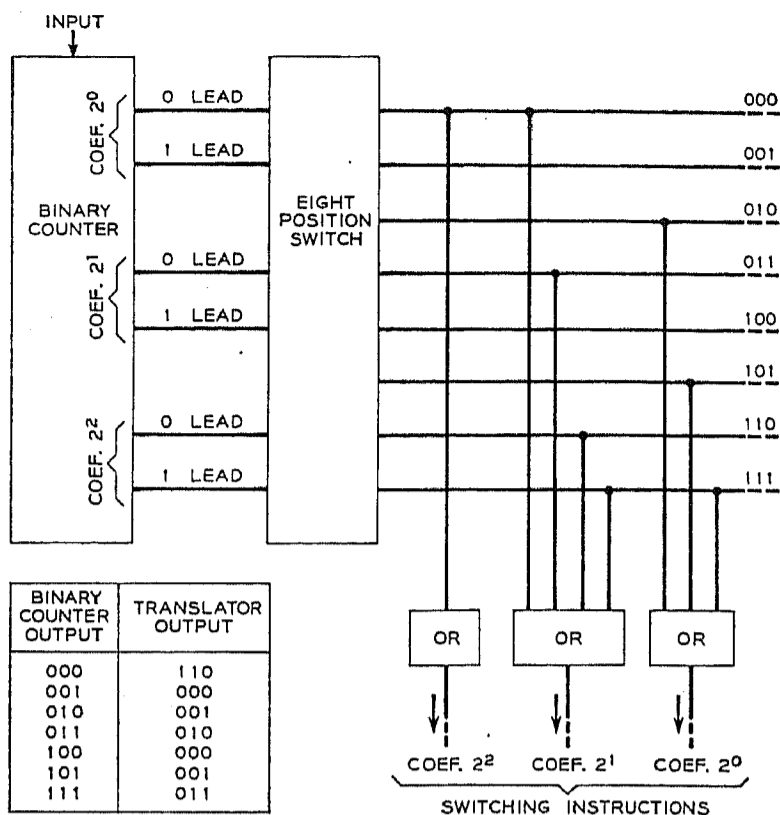


Figure 16 (left).  
Translator

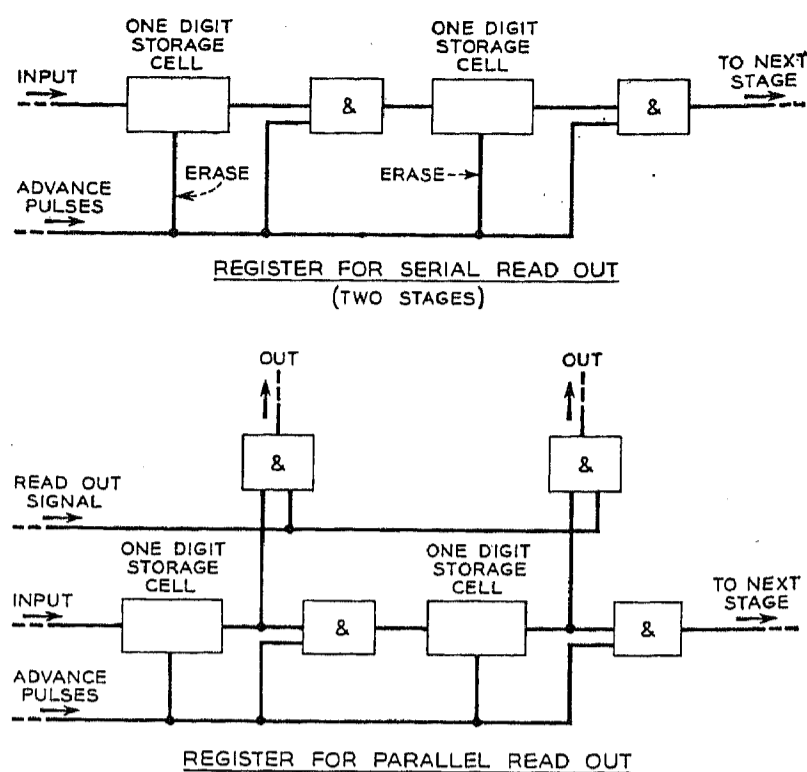


Figure 18 (right).  
Shift register

dividend to  $2w-1$  places, and then to form the complement of the divisor. The divisor complement is first added to the dividend. The  $2w-1$ 'th digit of the sum is noted. In the example, the digit is a 1, indicating a negative result. The machine then writes a zero in the quotient, drops out the  $2w-1$ 'th digit of the previous sum, and adds half the divisor (divisor shifted one place to the right). The  $2w-1$ 'th digit is examined again and is found to be a 1. Another zero is written in the quotient, the  $2w-1$ 'th digit is dropped out, and the divisor is again shifted and added. This time the  $2w-1$ 'th digit is zero, indicating a positive sum. A 1 is written in the quotient, and the complement of the divisor is shifted and added. This process is continued until the 5-digit answer 0.0110 is obtained. Note that the first operation was made with the binary points of the divisor and dividend lined up. In general, it would be necessary to start with the least significant digit of the divisor written under the most significant digit of the dividend.  $2w(2w-1)$  digit times are required for division in a general-purpose machine.

The divider block diagram is shown in Figure 15. The divisor is stored in delay line 1 and recirculates there, becoming available every  $2w-1$  digit times. The 1's complement of the divisor is taken by

the inhibitor circuit fed by  $DL1$  (delay line 1). Every  $2w-1$  digit times switch  $S2$  selects either the 1's complement of the divisor or the divisor itself to go into the adder. The switch selects the 1's complement when it is desired to subtract. However, to obtain subtraction it is necessary to add the 2's complement. This is done by bringing out the carry terminal of the adder, and whenever  $S2$  is thrown to select the 1's complement a 1 is sent into the carry terminal and, in effect, makes the augend the 2's complement.

The storage cell of switch  $S2$  receives an erase signal every  $2w-1$  digit times. At the same time, the switch receives a 1 or a zero from the inhibitor circuit on the adder output. If this inhibitor sends a 1 to the switch, the  $2w-1$ 'th digit of the sum just completed was a zero indicating a positive number and switch  $S2$  goes to position  $b$  to select the 1's complement of the divisor for the next addition. The output of the inhibitor circuit also is sent to the carry terminal of the adder, so that when  $S2$  is at  $b$  a 1 will be injected to convert the 1's complement to a 2's complement. Each output of the inhibitor is also a digit of the quotient, for if the  $2w-1$ 'th digit of the sum is a zero, a 1 should be written in the quotient and the inhibitor supplies that 1. It is written in delay line 3, which has a circulation time

of  $2w$ . Since a new quotient digit is received every  $2w-1$ 'th digit time, delay line 3 has as its output the digits of the quotient in reverse order to that in which they are obtained, which puts the least significant digits first as they should be.

The foregoing discussion shows how the augend is obtained for the adder. The addend is obtained from  $S1B$ . This switch selects the dividend during the first  $2w-1$  period, and thereafter selects the previous sum from delay line 2. Note that an inhibitor circuit between the adder and  $DL2$  blocks out the  $2w-1$ 'th digit each time to keep the remainder from growing longer than  $2w-1$ . Since delay line 2 has  $2w$  delay and delay line 1 has  $2w-1$  delay, the augend arrives one digit early each  $2w-1$  period with respect to the addend. This feature supplies the successive divisions by two that are required.

Delay line 4 exists merely to supply the  $2w-1$  pulse that is sent to  $S2$  and the inhibitor circuit on the adder output. Switch  $S1$  is driven to  $a$  at the end of the computation period to empty the delay lines of the dividend, divisor, and  $2w-1$  pulse preparatory to solving a new problem. After the first  $2w-1$  digit times of a new computation, all sections of  $S1$  are thrown to  $b$  and recirculate their data.

An estimate based on Tables I and II of the parts required by the divider of Figure 16 shows that to handle 48 digit numbers 371 crystal diodes, 390 digits of delay, and 51 active elements would be required. This assumes that the delay lines have their data regenerated every eight digits. If the data are regenerated only every 16 digits, only 36 active elements are required. The divider is seen to be of about the same complexity of the multiplier.

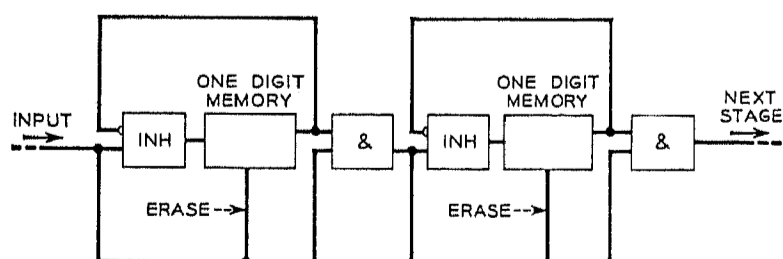


Figure 17. Binary counter

## Translator

This unit translates one set of numbers or words into another set. Storage may be regarded as translation in which addresses are translated into the words stored at the addresses.

Translators are useful in developing switching orders. At the completion of some process, it may be necessary to develop a new set of orders. When the orders must be developed in a certain sequence, a binary counter can be used to develop the input for the translator. Whenever the next set of orders are required, a pulse is fed to the counter. The binary count number is used as the switching instruction for a switch. Every time the counter receives an input pulse the switch will be advanced one position and will put a pulse on the selected lead. The various output leads of the switch will be connected to a group of or-circuits. One or-circuit will be required for each digit of the translator output. Figure 16 shows a scheme for generating eight sets of 3-digit words. To generate  $n$  digit words  $n$  or-circuits would be required. No more than eight leads would be required for any or-circuit. The translator shown develops the output word in parallel. If the or-circuits are fed to taps in a delay line, the output word will be received serially from the output of the delay line.

## Binary Counter

Figure 17 shows how each stage of a binary counter can be made from a 1-digit memory cell plus an inhibitor and an and-circuit. In Figure 17, if the count is at 01 and a 1 is sent into the counter, the 1 in the first stage will be admitted to the second stage. The 1 that was in the first stage inhibits the 1 input for the first

stage and the stage therefore goes to 0 and the count is 10. If another 1 is sent into the counter, it will enter the first stage and the count will advance to 11. If another 1 is fed in, the first two stages will set to zero and the third will receive a 1 making the count 100. This type of counter can be used only to count synchronized input pulses but that is all that a counter is expected to do in a serial computer.

In a serial machine binary counters often can be replaced by simpler devices. Suppose that it is desired to time some operation and know when 64 digit times have elapsed. A 6-stage counter could be used or a digit pulse could be circulated in a 7- and a 9-digit delay line. If an and-circuit were fed by the two delay lines, 63 digit times would elapse before the and-circuit were operated. If the output were fed through a 1-digit delay line the desired 64 digit delay would be obtained.

## Shift Registers

Shift registers are sometimes needed to take data at a low rate, from a magnetic drum, for example, and retransmit it at a rapid rate. A shift register that is loaded serially may be read out in parallel on  $W$  leads or in series on one lead. Each stage of such registers can be built around a 1-digit memory cell. In Figure 18 are illustrated two stages of each kind of register. The numbers in each cell are erased and advanced to the next cell whenever an advance pulse is received.

Advance pulses will be received every time a new digit is to be loaded. When a number is in the register, the advance pulses can be stopped and the number can be stored until it is required. In the case of the parallel-read-out registers, extra and-circuits are used at each stage. These are fed by the read-out signal.

Besides their use in changing the rate of transmission of data, the shift registers described can store words for integral multiples of a digit time. Where the storage period is an integral multiple of a word time it is more efficient to use delay lines as discussed previously. It may be mentioned that in a synchronous computer tapped delay lines can replace shift registers for the conversion of serial to parallel data and vice versa.

## Conclusions

A fairly extensive catalogue of digital computer components has been given in block diagram form with estimates of the number of parts they require. The designs have the common feature of using active elements as amplifiers only. The principal conclusion to be drawn is that an all-semiconductor computer can be built with diodes and transistors. This paper has been written as a contribution towards the development of such a computer.

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No Discussion

# Computing Circuits and Devices for Industrial Process Functions

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**P**OTENTIOMETER and bridge-balance systems using galvanometer detectors and electromechanical amplifiers have been used successfully in the recording of industrial process variables for many years. Electronic detection and amplification has greatly extended the field of application of these basic instruments by increasing response speed, sensitivity, and range of application.<sup>1,2</sup> The versatility and standardization of design made possible by the electronic servo amplifier has permitted the utilization of the accurate null-type metering system to the continuous measurement and control of practically all process variables which can be converted to electrical functions. The electronically operated null-balance recorder is admirably suited to perform as an analogue calculator, and circuits and devices are available which permit the computation of derived process variables including sums, differences, products, ratios, powers, roots, and trigonometric functions. The electric computing systems offer numerous advantages over equivalent mechanical methods. These include:

1. Transmitters (also called transducers or pickups) whose outputs are to be included in the calculation can be located remotely from one another in their most favorable location relative to process. In this way the electric system offers the additional advantage of short-range telemetering for centralized process control.
2. Complete electronically operated computing systems are built up using the basic standard components of the simple measuring systems. Mechanical computers generally require special design of linkages, cams, and gears.
3. Functions can be computed accurately over wide ranges and with relatively straightforward calibration procedure. Mechanical computers involving complex linkages and cams are generally limited and require special calibrations for each application.

Analogue-type computers using the servo operated systems are very well suited to the continuous recording of derived process functions. Digital computers and also analogue systems which are entirely electronic in operation do not appear, at least at the present, to be suitable for industrial process control. For this

service the requirements are sturdiness of design, the necessity for remote indication and recording of results, long life, reliability, and ease of service. The fundamental mathematical accuracy of the more complex computing machines usually is not required since the accuracy almost always is limited by the primary measuring element or the transducer which converts the physical quantity to an electrical function.

Some of the applications of servo actuated computers of the automatic null-balance type are:

1. Summation of fluid flows in chemical and other type industrial processes.
2. Computation of ratios of related factors as, for example, fuel-air ratio in combustion control and control of the ratio of chemicals to obtain a desired solution in a process.
3. Calculation of heat released or absorbed in heat exchangers and refrigeration systems.
4. Calculation of over-all plant efficiencies as, for example, that of a steam-electric generating station.
5. Accurate compensation of measurements as, for example, automatic correction of gas flow meters for wide range variations in the density of the gas.
6. Automatic and continuous solution of theoretical and empirical equations relating a number of process variables such as pressure, temperature, pH, conductivity, and flow to a final product output.

## The Elementary System

Figure 1 is a diagram of the automatic null-balance system of measurement and shows in block form the basic components of the system. The wide application of the null-balance system to the measurement and control of process variables is illustrated. In some applications it is possible to perform simple computations such as addition, subtraction, and division directly in the basic measuring circuit. For example, temperature difference can be computed by simply incorporating two temperature detectors in a bridge or potentiometer circuit in such a way that the balancing position of the follow-up is made directly proportional to the difference. In other applications, particularly where the energy output of the

measuring element is very small or where more complex calculations are required, retransmission from a primary instrument is employed.

If the pickup is a primary measuring device such as a thermocouple, resistance thermometer, or photocell, the physical variable is converted directly into an electrical output such as current, voltage, or resistance. The miscellaneous or "quality" variables such as pH, vacuum, conductivity, and gas analysis are best measured by electric or electronic primary elements. Consequently, the output of these elements can sometimes be used directly in computers without the intermediate mechanical link. On the other hand, quantities such as pressure, fluid flow, and liquid level are converted most frequently into mechanical factors such as force or motion by means of a mechanical metering element or link which may be a float, diaphragm, bellows, liquid column, or bourdon tube. This metering element in turn actuates a position transmitter or pickup device whose electrical output is proportional to the physical variable. For many straightforward algebraic functions this output can be combined with the output of similar elements to perform the desired computation. However, in some applications the motion of the metering element bears a nonlinear relation to the process variable, as for example: fluid flow is proportional to the square root of differential pressure drop across an orifice or flow nozzle. If such outputs are to be combined for totalizing and similar functions, the nonlinear quantity must be converted to an electrical factor proportional to the desired quantity. In these applications the function is first extracted in a primary null balance type of receiver and retransmitted as an electrical quantity proportional to the process variable.

The principles discussed are illustrated by a British-thermal-unit calculating system shown diagrammatically in Figure 2. It is desired to compute the rate at which heat is absorbed by the fluid flowing through the heat exchanger. Two temperature-sensing elements  $T_1$  and  $T_2$  are placed in the inlet and outlet side of the heat exchanger and connected into an automatically balanced differential bridge or potentiometer circuit so that the mo-

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tion of the motor driven follow-up is proportional to temperature difference. The balancing motor also actuates a retransmitter which may be a resistance slide-wire. Flow is measured by means of a mechanical-type element responsive to the differential pressure drop across an orifice or other type of primary element in the pipe. This pickup device converts differential pressure into a telemetered electric variable and the motion or position of the follow-up in the flow receiver is responsive to this differential. Since the flow is proportional to the square root of the differential, the measured function is converted to flow in the receiver by means of a properly shaped cam operating a retransmitter whose output is made proportional to the flow. The output of the flow retransmitter is then combined with the output of the temperature-difference receiver in a multiplying circuit in which the receiver moves proportional to the product of the retransmitted variables to produce an indication or recording of the rate of British-thermal-unit absorption. A more detailed description of a typical computing system is presented in a later section of this paper.

## Pickups and Their Circuits

It is not the purpose of this paper to describe in detail the wide variety of primary metering elements, transmitting, and pickup devices as well as the basic measuring circuits of the null type frequently used in potentiometer and bridge-balance instruments. These are either well known or are adequately described elsewhere in

the literature. Primary measuring elements may be grouped into two general categories; namely, the self-generating type in which a small amount of energy from the process is converted into an output voltage, and the transducer type in which the measured variable influences either in direct proportion or by some known relation the electrical characteristics of the pickup device. Examples of the first class are thermocouples, photo-cells of the barrier layer type, and pH electrodes. Examples of the transducer type are resistance temperature detectors, resistance wire strain gages, electrolytic conductivity cells, and heated filaments for gas analysis by thermal conductivity.

Of more specific interest in the computing systems are the position pickup devices which convert position or motion into an electrical output. The following types of position transmitters and function generators are used in the analogue computing systems.

1. The adjustable core transformer, sometimes called the differential transformer or linear induction potentiometer.
2. The resistance slide-wire used either as a variable resistance or as a rheostat potentiometer.
3. Induction potentiometers for converting angle of rotation to a proportional output voltage.
4. Sine-cosine electrical resolvers for converting angle of rotation to electric voltages proportional to sine and cosine functions.

The first two types of simple position pickups are used most extensively in the type of process function computers covered by this paper and will be described in detail. Both of these devices may be used either as transmitters directly coupled to mechanically operated

metering devices or as retransmitters for generating voltages proportional to a function of the original variable. They are also used as servo-motor operated follow-ups in intermediate and final recording receivers.

## ADJUSTABLE CORE TRANSFORMER

Figure 3 illustrates the principles and application of the adjustable core transformer which is used extensively in the measurement of factors which are convertible to position or motion. The magnetic core of the transmitter is moved by the mechanical metering element which in some cases actually may be a complete meter which indicates or records the variable and in others only a basic link between the primary detecting element and the transmitting unit. Figures 3(A) and 3(B) show the structure and principle of operation. The magnetic flux linkages between the primary winding and each of the two secondaries is determined by the magnetic core position. Consequently the voltages induced in the secondary windings vary linearly and differentially in opposite directions in accordance with the following equations.

$$\begin{aligned} E_1 &= E_0 + kx \\ E_2 &= E_0 - kx \end{aligned} \quad (1)$$

where

$E_0$  = voltage when the core is in the exact center position  
 $x$  = per-unit motion of core = actual motion divided by total motion  
 $k$  = total change in voltage per winding at the extreme of core position

In the ratio-balance circuit of Figure 3(C) the secondaries are connected so that their voltages aid. At balance the

## Process Variables

### Measurements with d-c Circuits

TEMPERATURE  
pH  
SPEED  
VACUUM  
VISCOSITY  
GAS ANALYSIS  
(BY RADIATION  
ABSORPTION)  
ETC.

### Measuring Element

THERMOCOUPLE  
PHOTO CELL  
TACHOMETER  
ION GAGE  
ETC.

### Measurements with a-c Circuits

TEMPERATURE  
PRESSURE  
FLOW  
LEVEL  
POSITION  
ELECTRICAL  
CONDUCTIVITY  
GAS ANALYSIS  
SMOKE DENSITY  
WEIGHT  
STRESS  
ETC.

RESISTANCE  
THERMOMETER  
ELEMENT  
ADJUSTABLE CORE  
TRANSFORMER  
BOLOMETER  
ETC.

Figure 1. Block diagram of the automatic, null-balance system for measurement, computation, and control

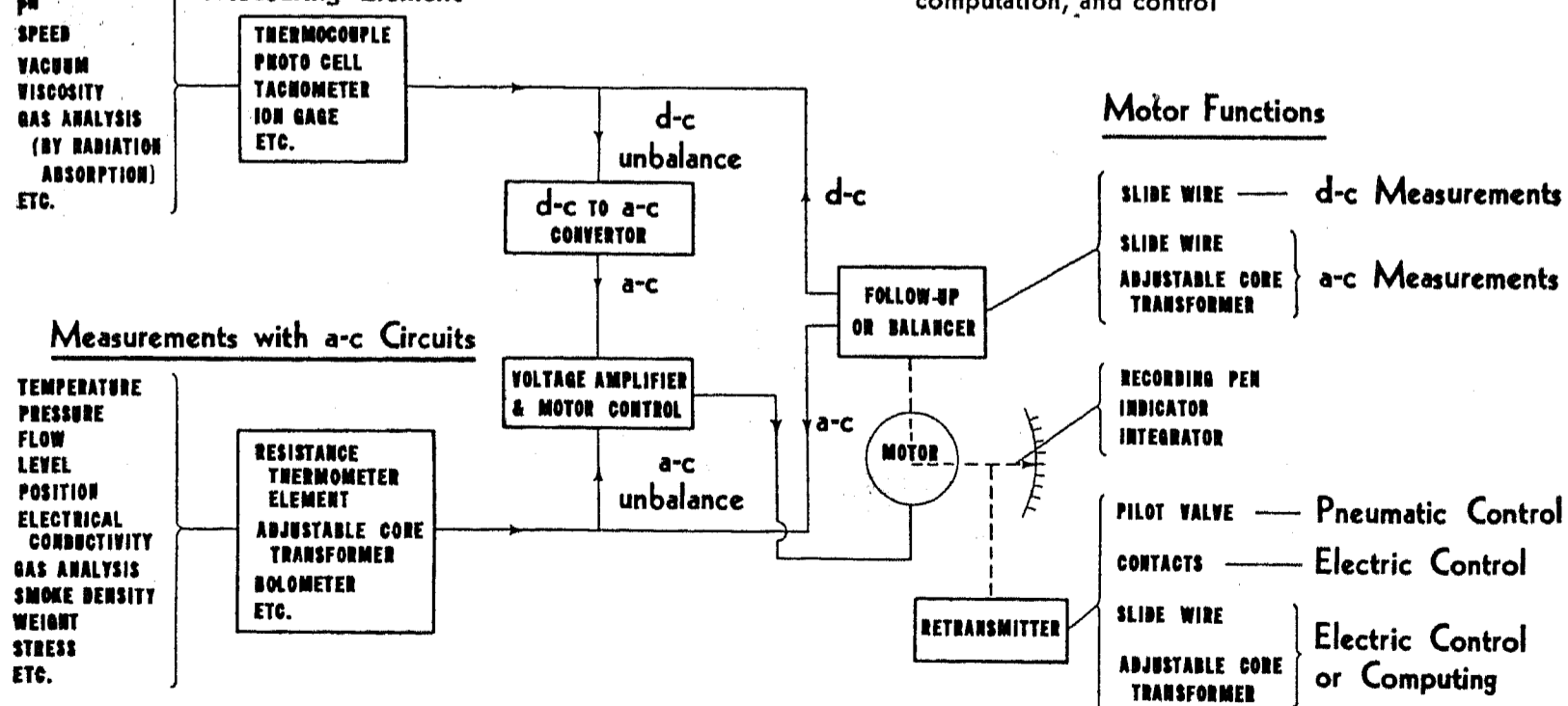
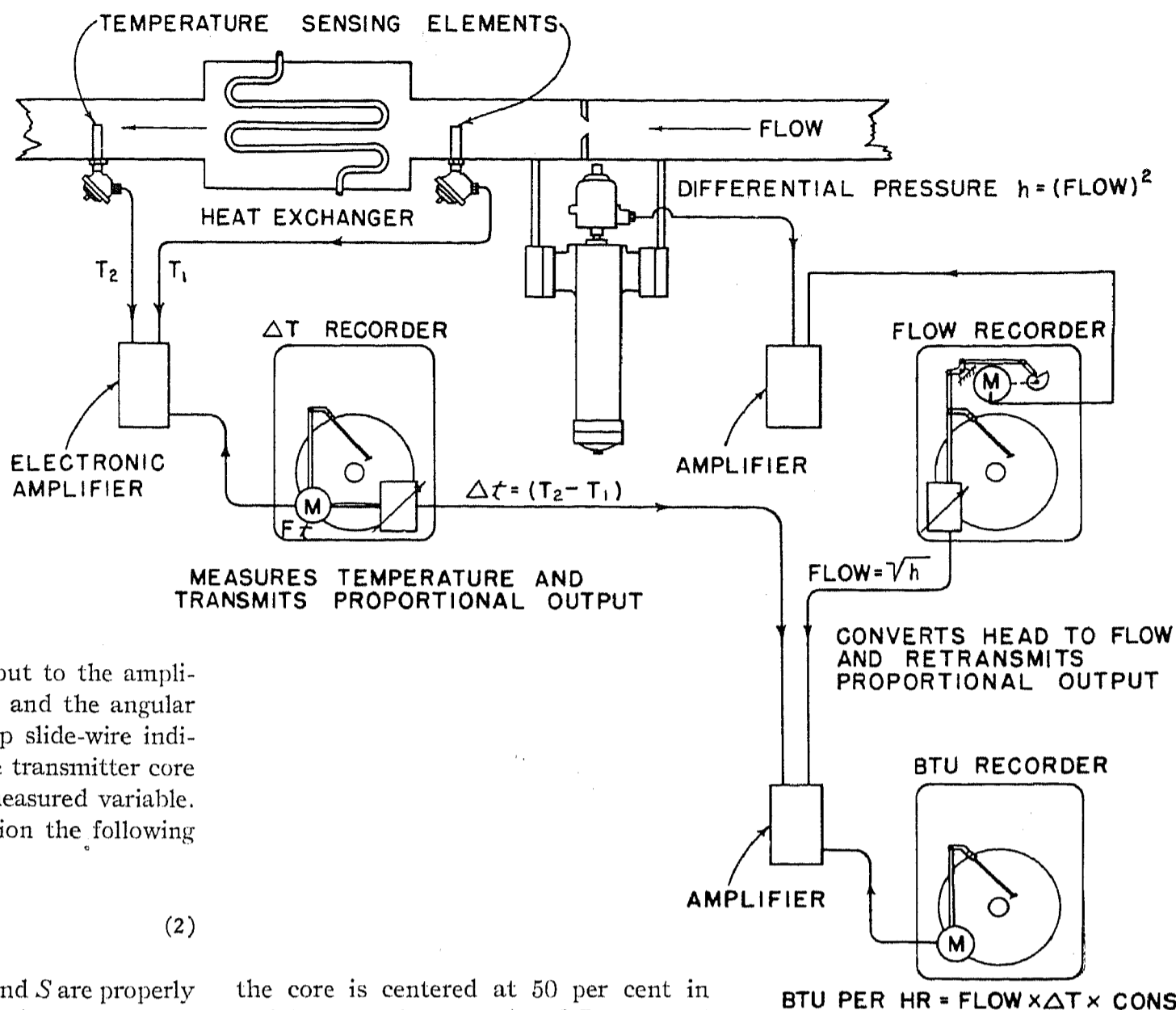


Figure 2. Diagram of a typical process computing system—British-thermal-unit calculator



fundamental voltage input to the amplifier is 0 or a minimum, and the angular position of the follow-up slide-wire indicates the position of the transmitter core and consequently the measured variable. For this balance condition the following equation applies.

$$\frac{E_0 + kx}{E_0 - kx} = \frac{B + MS}{A + (1 - M)S} \quad (2)$$

If the values of  $A$ ,  $B$  and  $S$  are properly chosen so that  $B = (E_0 S)/k$  and  $A = S(E_0/k - 1)$ , the slide-wire rotation  $M$  will equal the core position  $x$ . In this analysis it is assumed that the magnetic core is in the center of the coil when the transmitted variable is zero. In many applications

the core is centered at 50 per cent in which case resistances  $A$  and  $B$  are equal or  $A = B = S[E_0/(k - 1/2)]$ .

In the potentiometric circuit shown in Figure 3(D) the secondary windings of the transmitter are connected in phase opposition so that the total voltage is a pro-

portional function of the core position. The identical transformer unit in the receiver is positioned by the motor and the system is balanced when the transmitter and the receiver core positions are ex-

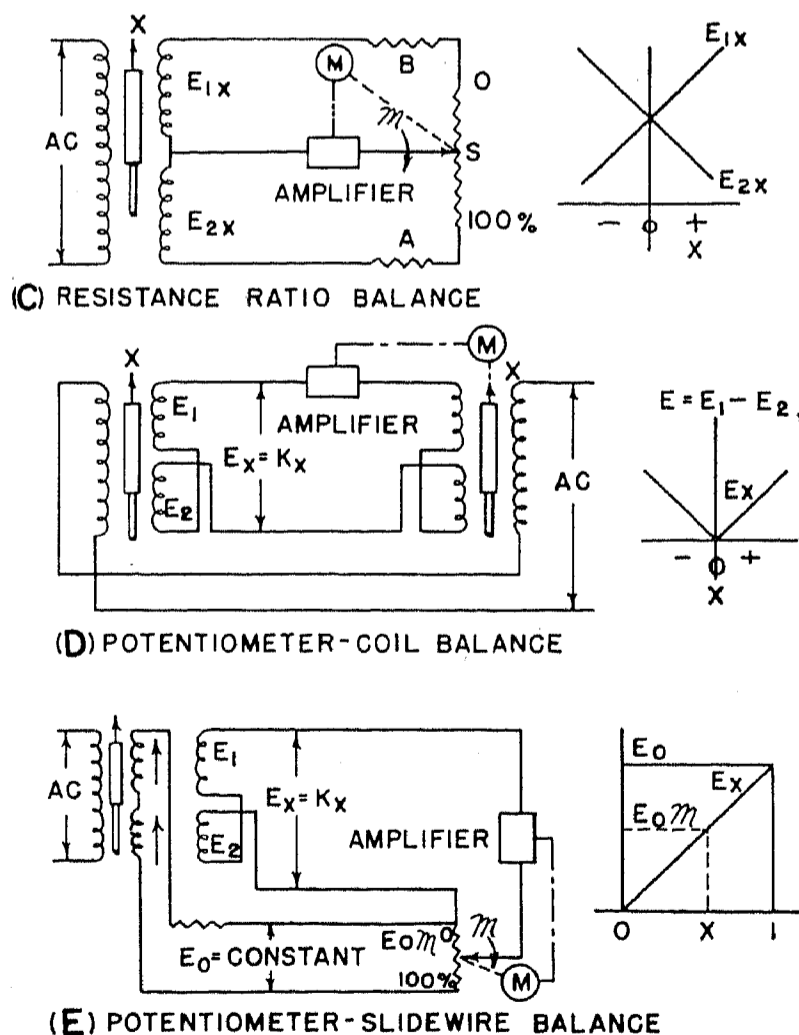
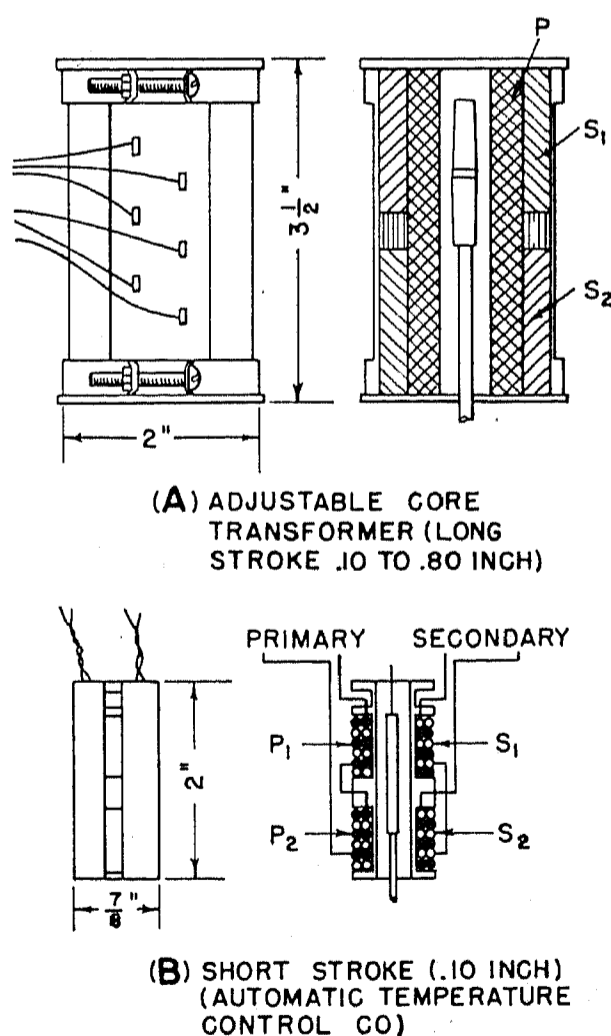


Figure 3. Basic measuring circuits using adjustable core transformer-type transmitters

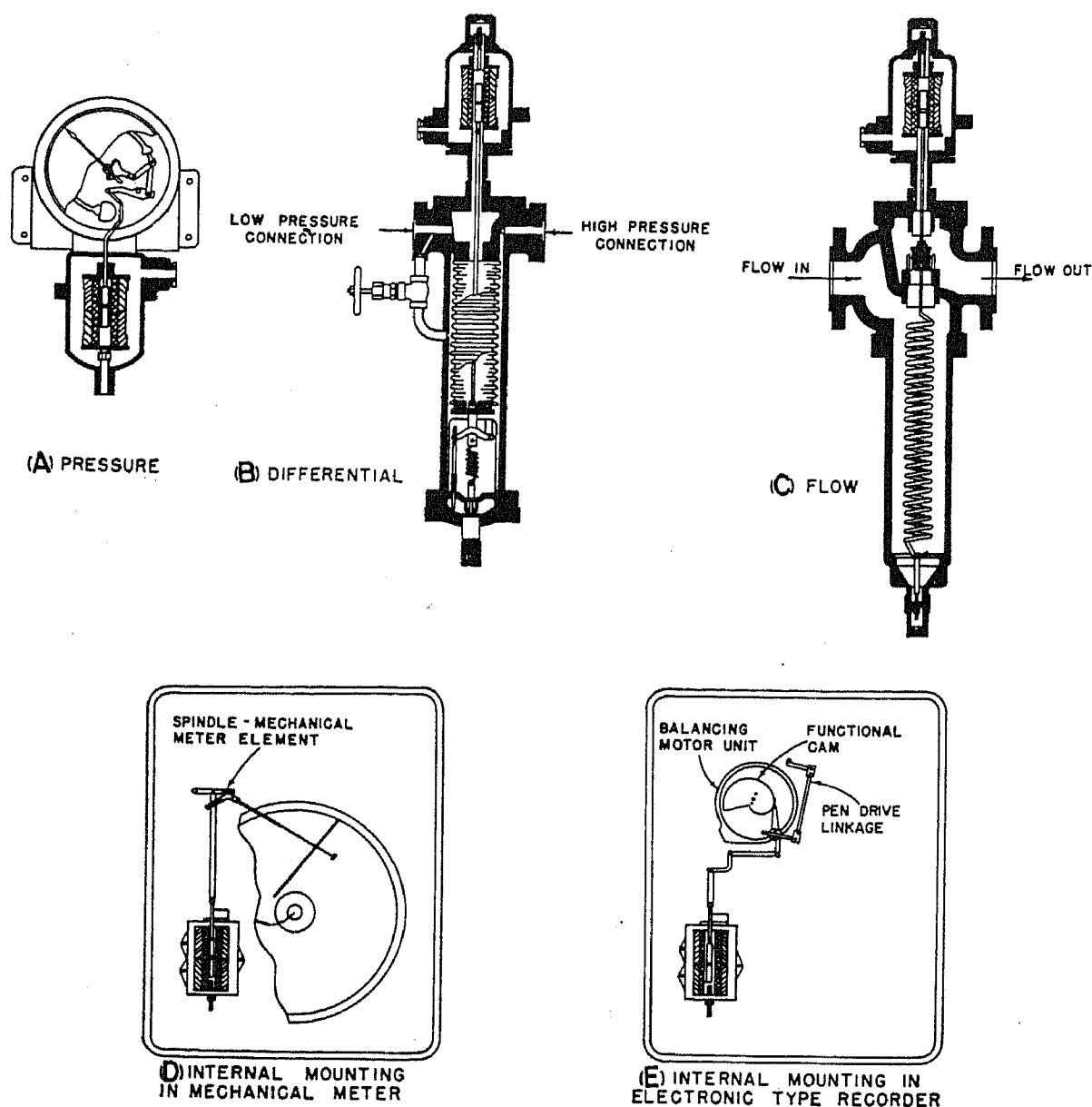


Figure 4. Pickups of the adjustable core transformer type for specific applications

actly the same. This system has several advantages over the one shown in Figure 3(C), the most important being the fact that no current flows in the leads of the measuring circuit. Consequently, it can be used over greater distances without being affected by line resistance.

In the slide-wire balance type of potentiometer system shown in Figure 3(E), two pairs of secondary windings are used on the transmitter. One pair with the secondary windings connected opposing sets up a voltage proportional to the transmitter core position. The other pair of windings connected with the voltages aiding sets up a reference potential for the balancing slide-wire. The system is calibrated by adjusting the voltage across this slide-wire by means of the series resistance so that  $E_0$  is equal to the maximum output of the transmitter.

A number of typical transmitters using the transformer pickup device are illustrated by Figures 4(A) to 4(E) inclusive. Figures 4(A), 4(B), and 4(C) show devices of the integral construction type in which the transformer unit is designed as a basic part of the mechanical measuring device. This type of pickup is particularly adaptable to the telemetering of flow and differential pressures where the

moving element of the basic meter device including the transformer core can be enclosed entirely in a high-pressure chamber.

In Figures 4(D) and 4(E), the transformer is incorporated as a retransmitting element in recording meters. Figure 4(D) shows the transformer transmitter operated through suitable linkage from the spindle of a mechanical flow meter and it is used in preference to the integral devices previously described where the core stroke is to be adjustable for calibration purposes. Figure 4(E) shows the core of the transformer actuated by the motor-driven follow-up of an electrically balanced recorder. This system is particularly advantageous where the output of the retransmitter must be a non-linear function of the measured variable. In a complex process computing system, many of these retransmitter converters might be used to order to translate the various metered functions such as pressure, temperature, and flow into electrical quantities.

#### RESISTANCE SLIDE-WIRE CIRCUITS

Resistance slide-wires are used either as pickups in the primary measurement

devices or as retransmitters in electronic and mechanical type meters, similarly to the adjustable core transformer already described. Since the transmitter may be situated in an unattended location, the slide-wire lacks the inherent sturdiness of the transformer device. Nevertheless, it is quite useful particularly where the computing system requires a resistance function. The friction of the sliding contact as compared with the essentially frictionless "floating action" of the movable core unit is also a disadvantage, particularly if the slide-wire is actuated by a low-power device such as a Bourdon tube pressure element.

Figure 5 illustrates the most commonly used null metering circuits of the bridge and potentiometric type which are used with the resistance slide-wire transmitter. These circuits are basically very simple and well known so that no detailed analysis of their characteristics is required. For algebraic calculations of process variable combinations the circuit shown in Figure 5(D) is particularly useful. In systems using this basic a-c potentiometer or ratio principle, the slide-wires function as sources of voltage proportional to the transmitted variables. The advantages of the potential balance circuits over Wheatstone bridge circuits perhaps are fairly obvious, but the most important ones are:

1. Elimination of contact resistance errors.
2. Relative ease of calibration and adjustment to take care of wide ranges in magnitudes.

Figure 6(A) shows the low-torque slide-wire transmitter for operation directly from a mechanical metering element. Because of its low friction, it can be geared or linked directly to the output of a sensitive metering element for converting the physical variable to an electrical output. Figure 6(B) shows the internal mounting of this element in a recording mechanical meter. If no record is required at the transmitting end, a pickup of the type shown in Figure 6(C) can be used in which the low-torque slide-wire is incorporated as an integral part of the mechanical device. Figure 6(D) shows the motor-driven slide-wire unit in the electronically balanced recorder. One slide-wire of this assembly is used to balance the primary metering circuit and the second slide-wire is used as a retransmitter for computing purposes. In some applications the adjustable core transformer is also used as a part of this assembly either as a follow-up in a basic metering circuit or as a retransmitter as shown in Figure 5(E).

## Basic Computing Circuits

### TRANSFORMER TYPE

Figure 7 shows the elementary circuits using the transformer transmitter for the simple arithmetic functions of addition, subtraction, and division of transmitted variables.

In the circuit of Figure 7(A) the per-unit rotation of the motor-driven balancing slide-wire is equal to the total or average motion of all the transmitter cores. The lower or zero end windings all are connected in series aiding so that the total voltage of one leg of the bridge is given by

$$E_2 = E_{x2} + E_{y2} + E_{z2} + \dots E_{n2} \quad (3)$$

Similarly, all the top windings are connected in series aiding so that

$$E_1 = E_{x1} + E_{y1} + E_{z1} + \dots E_{n1} \quad (4)$$

at balance

$$\frac{E_1}{E_2} = \frac{E_{x1} + \dots E_{n1}}{E_{x2} + \dots E_{n2}} = \frac{B + MS}{A + S - MS} \quad (5)$$

where

$E_{n1}$  = voltage output of the top winding of the  $N$ th transmitter

$E_{n2}$  = voltage output of the bottom winding of the  $N$ th transmitter

If the cores are centered at 0.5 of maximum stroke

$$\begin{aligned} E_{x1} &= [E_0 - (k/2)] + kx \\ E_{n1} &= [E_0 - (k/2)] + kn \\ E_{x2} &= [E_0 + (k/2)] - kx \\ E_{n2} &= [E_0 + (k/2)] - kn \end{aligned} \quad (6)$$

Substituting these values of voltages in the balance equation 5 yields

$$E_1 = \frac{N[E_0 - (k/2)] + k(x + \dots n)}{N[E_0 + (k/2)] - k(x + \dots n)} = \frac{B + MS}{A + S - MS} \quad (7)$$

If the fixed resistors  $A$  and  $B$  are adjusted equal so that

$$A = B = S[E_0/k - (1/2)] \quad (8)$$

The slide-wire rotation  $m$  is given by

$$M = \frac{x + \dots n}{N} \quad (9)$$

$n$  = motion of the  $N$ th transmitter

where  $N$  = the total number of transmitters.

For example, in Figure 7(A) there are three variables  $X$ ,  $Y$ , and  $Z$ . Consequently

$$M = \frac{X + Y + Z}{3} \quad (10)$$

If the ranges of the transmitters are not equal, the strokes of the cores are ad-

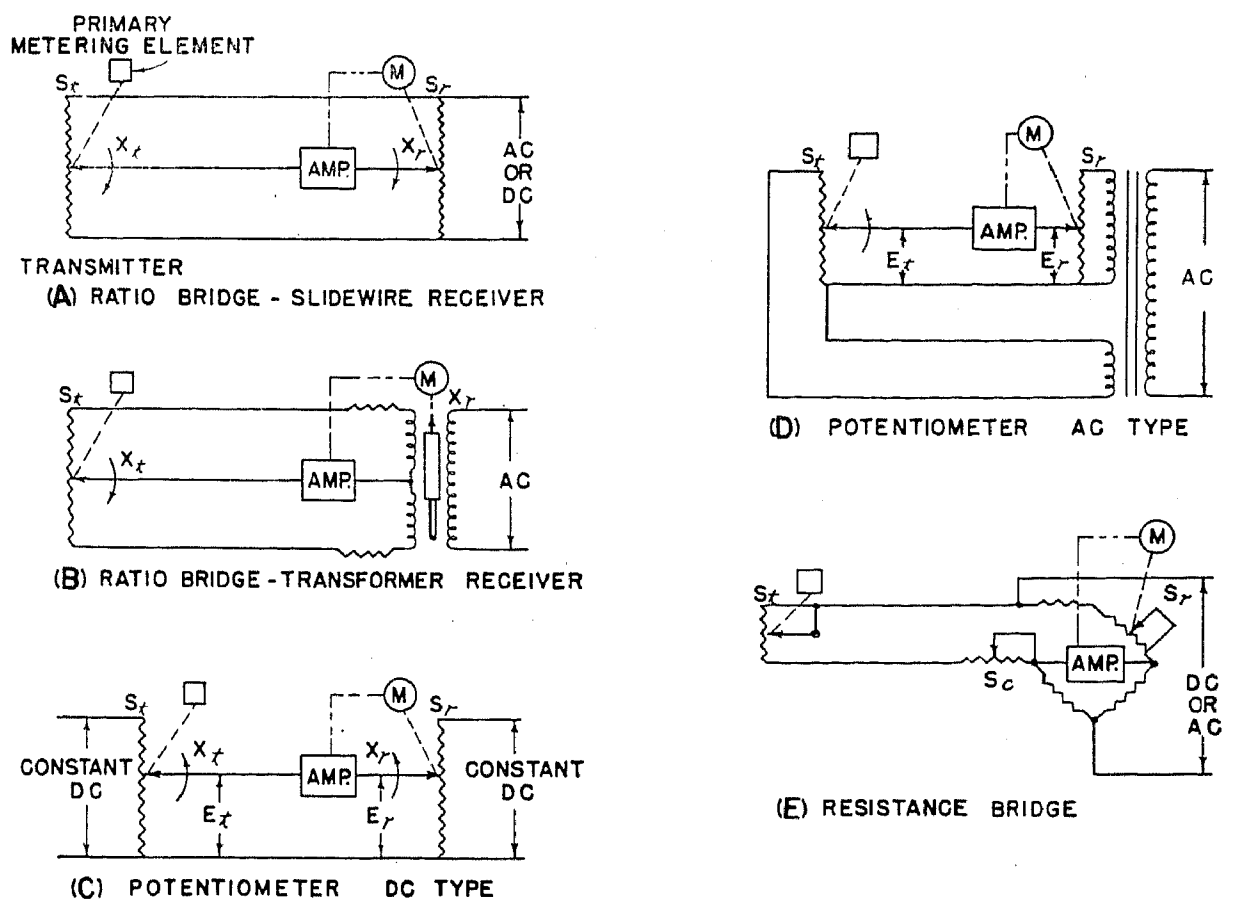


Figure 5. Basic circuits of the resistance slide-wire

justed proportionally.

Subtraction is accomplished in a similar manner as shown in Figure 7(D) by connected opposite secondary windings in series. The difference between the totals of two groups of variables can be accomplished by properly interconnecting the secondary windings.

In the totalizing and subtracting circuits shown by Figures 7(B) and 7(D) secondary windings are connected in phase opposition to produce voltages proportional to each transmitted variable.

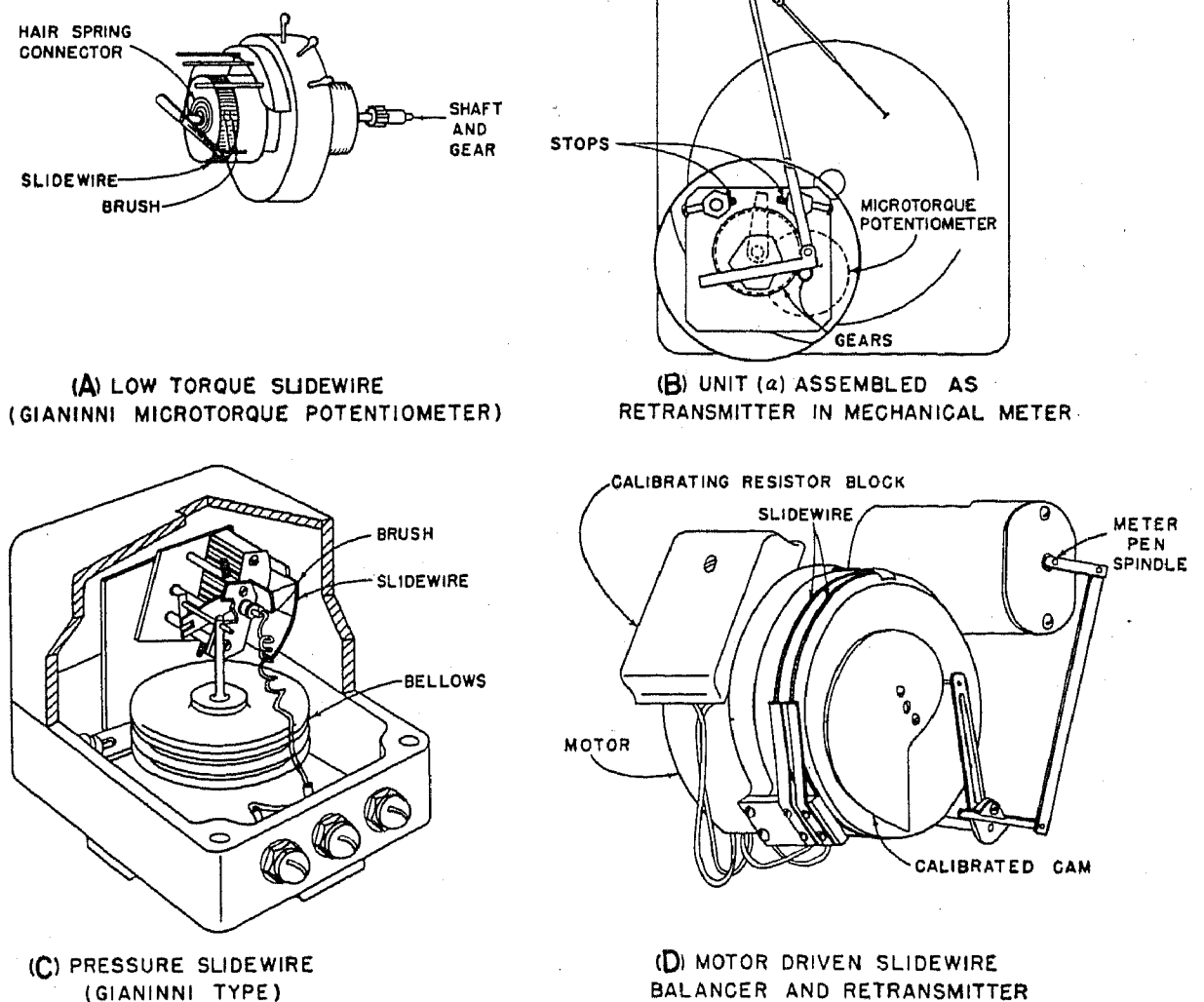


Figure 6. Pickups of the slide-wire type for specific applications

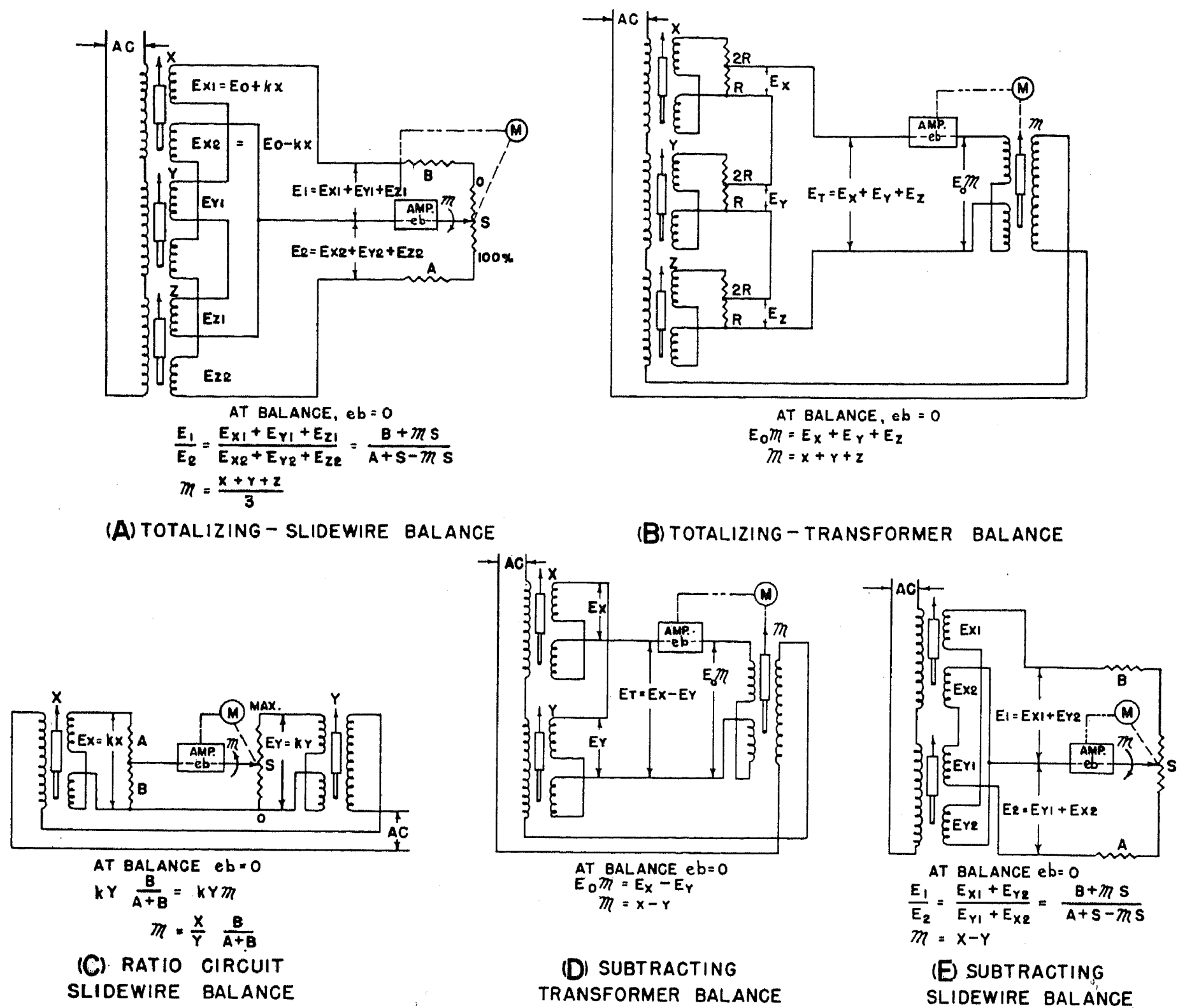


Figure 7. Basic computing circuits using adjustable core transformers

The balancing follow-up is a similar transformer unit. The proportion of the voltage of each transmitter is adjusted by means of potential dividing resistors to the full-scale range. In Figure 7(B) it is assumed that all ranges are equal.

In the ratio computing circuit shown by Figure 7(C) a fixed proportion of the linear voltage output of transmitter  $x$  is compared against an adjustable proportion of the output of transmitter  $Y$ . The proportion of the  $Y$  output required to balance depends on the position  $M$  of the balancing slide-wire.  $A$  and  $B$  are calibrating resistors determined by the range in ratios to be measured.

#### SLIDE-WIRE TYPE

The elementary computing circuits most frequently used with slide-wire

transmitter and receiver elements are shown in Figure 8. These circuits are all of the a-c potentiometer\* type and are generally preferable to Wheatstone bridge circuits because of: elimination of effect of slide-wire brush contact resistance, and wide flexibility and ease of calibration using a minimum number of fixed elements. Multiwinding transformers of shielded construction having negligible phase shift are used to supply proportional voltages to each computing slide-wire.

Totalization and subtraction are accomplished simply by algebraic addition of the voltage outputs of each transmitting slide-wire and comparing this total output against that developed by the

balancing slide-wire in the receiver. Multiplication and division of transmitted functions are accomplished as shown in Figures 8(A) and 8(B) by connecting the computing slide-wires in cascade. To avoid the error produced by the loading effect of the second slide-wire, it is necessary that its resistance be high relative to the first. The deviation from linearity of the voltage output of the second slide is determined by the following analysis.

The voltage across the second slide-wire is given by

$$e = \frac{E_0 A M}{A + M - M^2} \quad (11)$$

$E_0$  = supply voltage  
 $M$  = per unit rotation of first slide-wire  
 $A = S_2/S_1$  where  $S_2$  and  $S_1$  are the resistances of the first and second slide-wire respectively.

\*Note that these circuits are only pseudopotentiometric since they are basically ratio types and do not measure a-c voltages except in terms of a percentage or portion of the supply voltage.

The theoretical linear voltage across the second slide-wire assuming its resistance to be infinitely large is

$$e_t = E_0 M \quad (12)$$

The deviation from linearity is given by

$$\Delta e = \frac{e_t - e}{E_0}$$

or

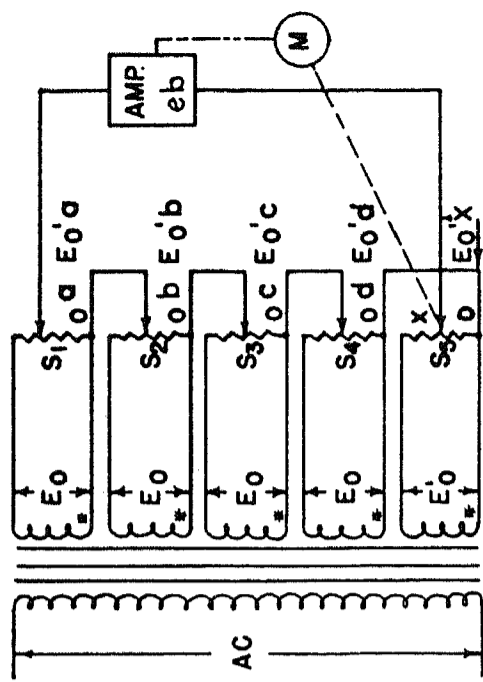
$$\Delta e = \frac{M^2(1-M)}{A+M(1-M)}$$

$$(13) \quad \text{For large values of } A \text{ this error is a maximum when}$$

$$M \cong 0.67$$

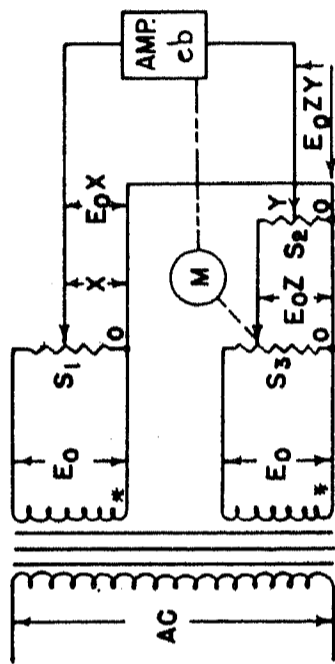
so that

$$(14) \quad \Delta e \text{ max} \cong 0.15/A \quad (15)$$



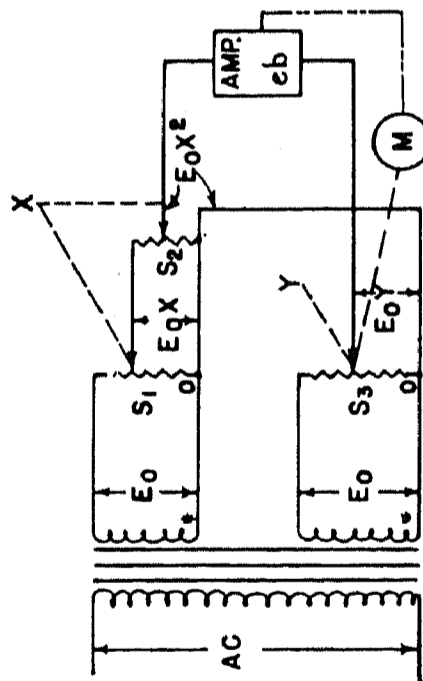
AT BALANCE  $eb = 0$   
 $E_0(a+b+c+d) = E_0'X$   
 IF  $E_0' = 4E_0$ ,  $X = \frac{a+b+c+d}{4}$

(A) MULTIPLICATION



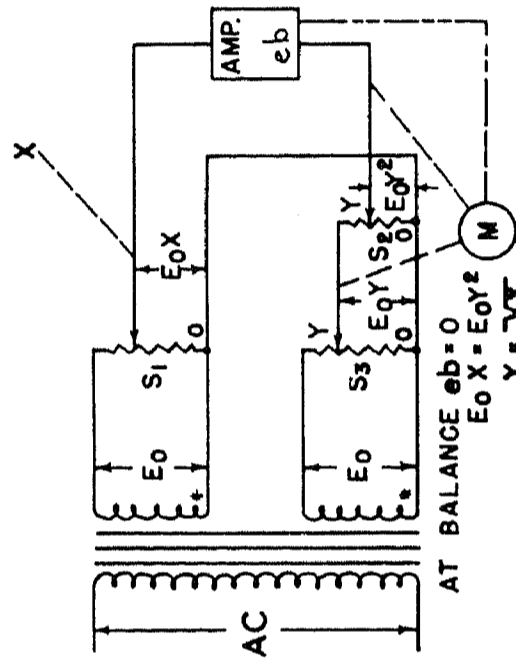
AT BALANCE  $eb = 0$   
 $E_0ZY = E_0X$   
 $Z = \frac{X}{Y}$

(B) DIVISION



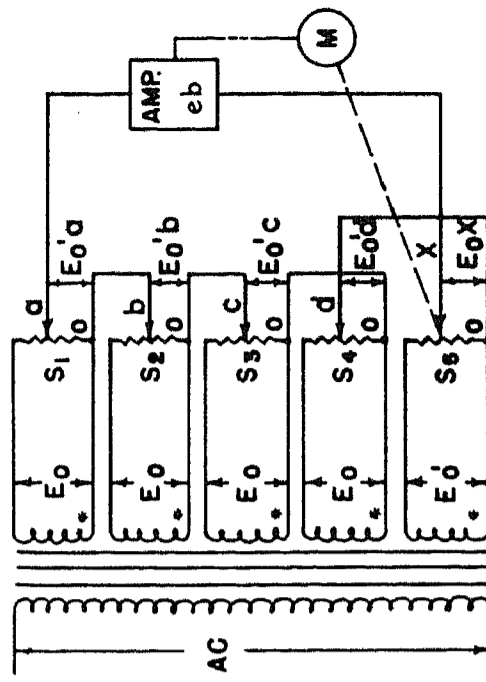
AT BALANCE  $eb = 0$   
 $E_0Y = E_0X$   
 $Y = X^2$

(D) SQUARING CIRCUIT



AT BALANCE  $eb = 0$   
 $E_0X = E_0Y^2$   
 $Y = \sqrt{X}$

(E) SQUARE ROOT



AT BALANCE  $eb = 0$   
 $E_0(a+b+c-d) = E_0'X$   
 IF  $E_0' = 3E_0$ ,  $X = \frac{a+b+c-d}{3}$

(F) TOTALIZING AND SUBTRACTING

\* INDICATES WINDING HAVING  
 HAVING SAME POLARITY.

Figure 8. Basic computing circuits using slide-wires

$$\Delta e_{\max} = 0.0015 = 0.15 \text{ per cent}$$

A voltage proportional to the square of the measured function is obtained simply by actuating two cascaded slide-wires from the transmitter as shown by Figure 8(D). Consequently at balance the output and rotation of the motor-driven slide-wire is equal to the square of the transmitted variable.

The square root of a function is obtained by using the circuit shown in Figure 8(E) in which the balancing motor operates simultaneously two cascaded slide-wires. Consequently, the voltage output of the receiver is proportional to the square of the motor shaft rotation. At balance this voltage must equal the linear voltage output of the transmitter so that the motion  $Y$  of the receiver is proportional to the square root of the measured variable.

The system shown by Figure 9 has proved to be a powerful tool in metering single variables or combinations of proc-

ess function where: the transmitter motion is a nonlinear function of the measured variable, and the measured variable is to be converted into another function for computing purposes. The relation between the balancing core motion  $Y$  and the servo motor shaft rotation  $M$  depends on the shape of the  $Y$  cam. Consequently  $M = F(Y)$ . ( $F$  denotes the function specified by the shape of the cam.)

$$M = F(x) \quad (16)$$

Since the motor shaft rotates in accordance with the desired function, a linear rise cam is used to actuate the linkage positioning the indicating, recording, or integrating elements of the receiver. Re-transmitting slide-wires actuated by the same motor reproduce the desired function as linear voltages or resistances for control, computations, or further telemetering.

As demonstrated by the figure, this system is frequently used to convert a measurement of differential pressure, such as would be obtained by the transmitter of Figure 4(B), to fluid flow for linear recording. In other applications this system is used to convert a measured variable such as temperature to some nonlinear function of this variable such as the specific heat of a heat transfer fluid.

If a slide-wire is used as the transmitter, the basic circuit of Figure 3(C) is inverted using a double cam, transformer balance follow-up as in Figure 9.

The cam-operated transformer balancer is generally superior to the tapered slide-wire follow-up for the conversion of functions because: cams can be cut to a greater degree of accuracy so that slide-wires can be wound to the desired func-

tions, and the range of taper of slide-wires is relatively limited whereas the cam rise rate can be made to cover a range from zero to almost infinity.

## A Typical Computer—Density Compensated Gas Flow Meter

With the components and elementary circuits already described, systems of any desired complexity can be designed to perform a specific computation. An important application of these to a system for recording pressure, temperature, and compensated gas flow is illustrated by Figure 10. The complete elementary circuit for this metering system is shown in Figure 11.

Equation 17<sup>4</sup> gives the weight rate of flow as a function of the differential pressure drop across the primary metering element.

$$W = 358.9 \text{ Cf} D_2^2 \sqrt{h\rho} \quad (17)$$

$W$  = flow rate, pounds per hour

$D_2$  = orifice diameter, inches

$h$  = differential pressure drop, inches of water at 68 degrees fahrenheit

$\rho$  = density of flowing fluid, pounds per cubic foot

$C$  and  $f$  are design constants

For an ideal gas

$$\rho = \rho_s \frac{P}{P_s} \frac{T_s}{T} \quad (18)$$

$\rho_s$  = density at standard or design conditions

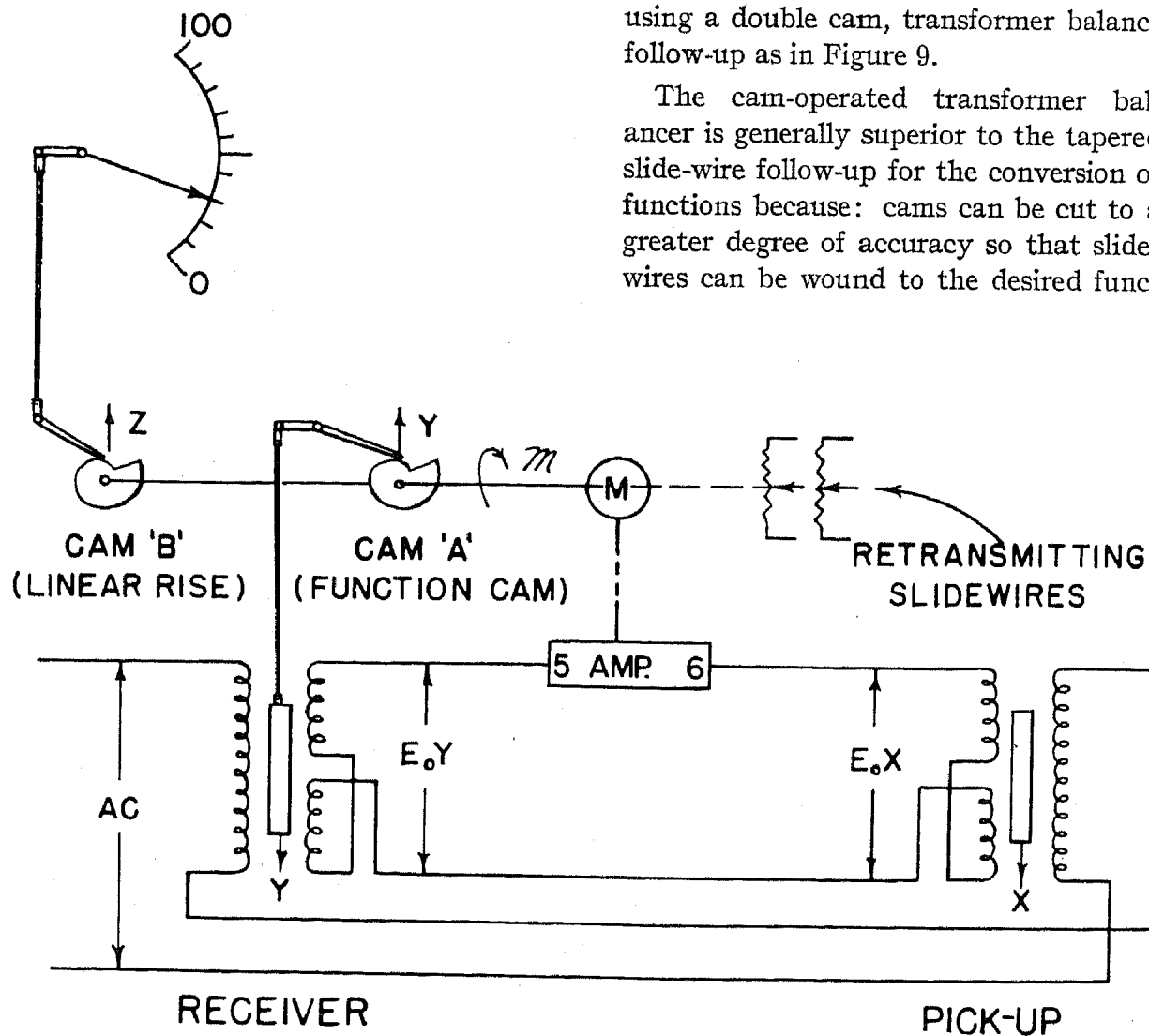
$P_s, T_s$  = design pressure and temperature

$P, T$  = operating pressure and temperature

Substituting equation 13 in equation 12 yields

$$W = 358.9 C_f D_2^2 \sqrt{h P_s \frac{P T_s}{P_s T}} \quad (19)$$

Combining constants in equation 14



**Figure 9. Nonlinear function computer**

AT BALANCE,  $E_o Y = E_o X$   
 $Y = X$

IF Y (RISE OF CAM 'A') =  $m^a$   
(MOTOR SHAFT ROTATION)

$$m = Y^{1/a} = X^{1/a}$$

$Z = M = X \frac{1}{2} =$  CHART OR SCALE  
READING.

### EXAMPLE

IF TRANSMITTER IS ACTUATED  
BY DIFFERENTIAL PRESSURE  
AS BY A BELLOWS TYPE  
METER.

$$X = h = \text{CONSTANT} \cdot (\text{FLOW})^2$$

MAKE  $Y = \eta^2 = X$

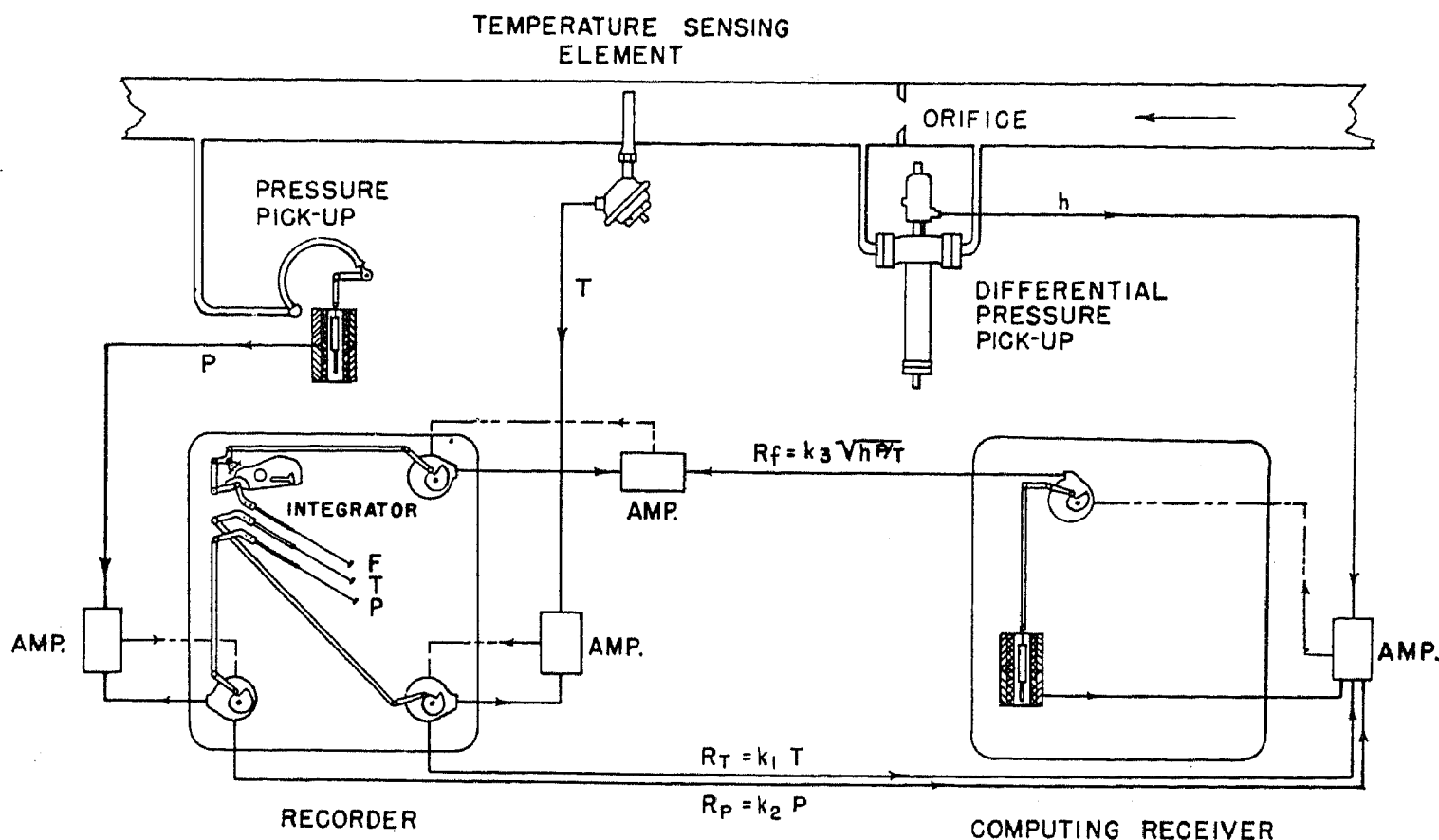
THEN  $m = \sqrt{Y} = \sqrt{h} =$

CONSTANT - FLOW.

**Figure 10. Schematic diagram of density compensated gas flow rate meter**

Recorder records and transmits pressure and temperature, records compensated flow, and integrates compensated flow

Computing receiver computes and transmits compensated flow



$$W = K_3 \frac{\sqrt{hP}}{T} \quad (20)$$

$$Y = \frac{hP}{T}$$

The system is essentially an analogue computer which solves equation 20 by converting  $P$ ,  $T$ , and  $h$  to electrical quantities and extracting the square root with the cam positioning nonlinear computer of Figure 9.

The circuit of Figure 11 is balanced when the voltage developed across the pressure compensating slide-wire is equal to the voltage across the temperature slide-wire.

$$KYT = KXP = KhP \quad (21)$$

and the position of the balancing transformer core is given by

If the cam which positions the core is shaped so that its rise is given by

$$Y = M^2$$

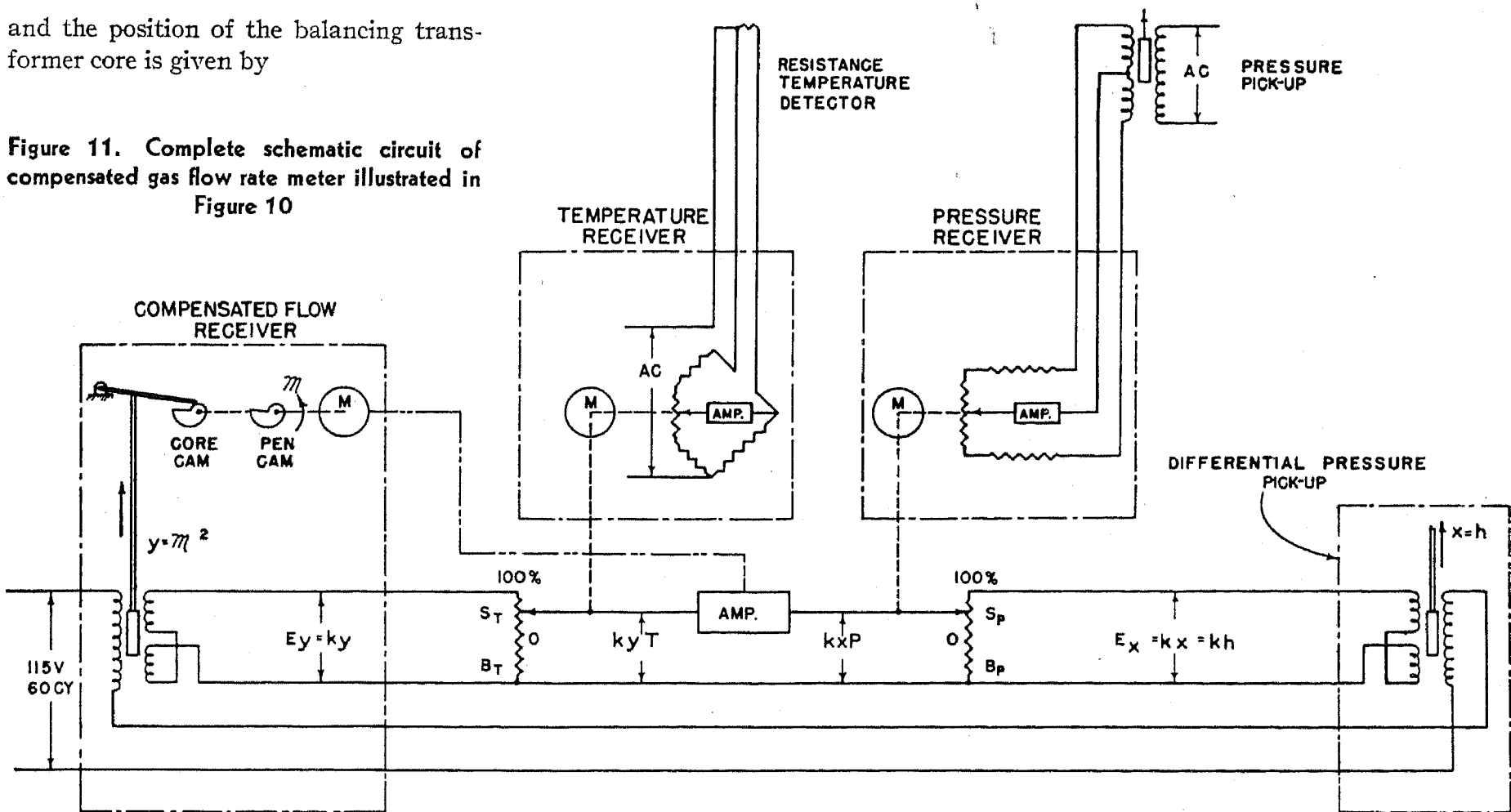
$$M = \sqrt{Y} = \frac{\sqrt{hP}}{T} = W \quad (22)$$

Consequently the angular position  $M$  of the servo motor shaft is a linear function of compensated flow

The specific arrangement of Figure 10

illustrates how the basic elements can be arranged to suit the particular metering requirements. In this system correlating records of pressure, temperature, and compensated flow as well as an integration of flow are required in a single meter. Space limitations require that the computing receiver employing the cam positioned transformer follow-up be placed in a separate case. Compensated flow is retransmitted from this computing unit to the recorder by means of a retransmitting slide-wire and corresponding receiving unit. If the integrator were not required,

**Figure 11. Complete schematic circuit of compensated gas flow rate meter illustrated in Figure 10**



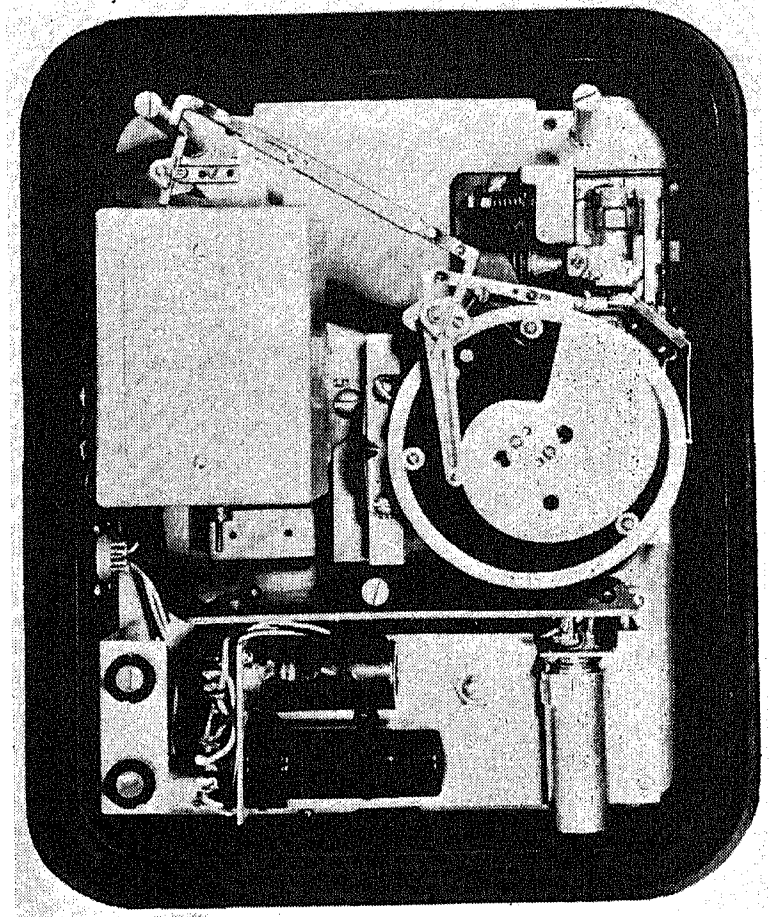
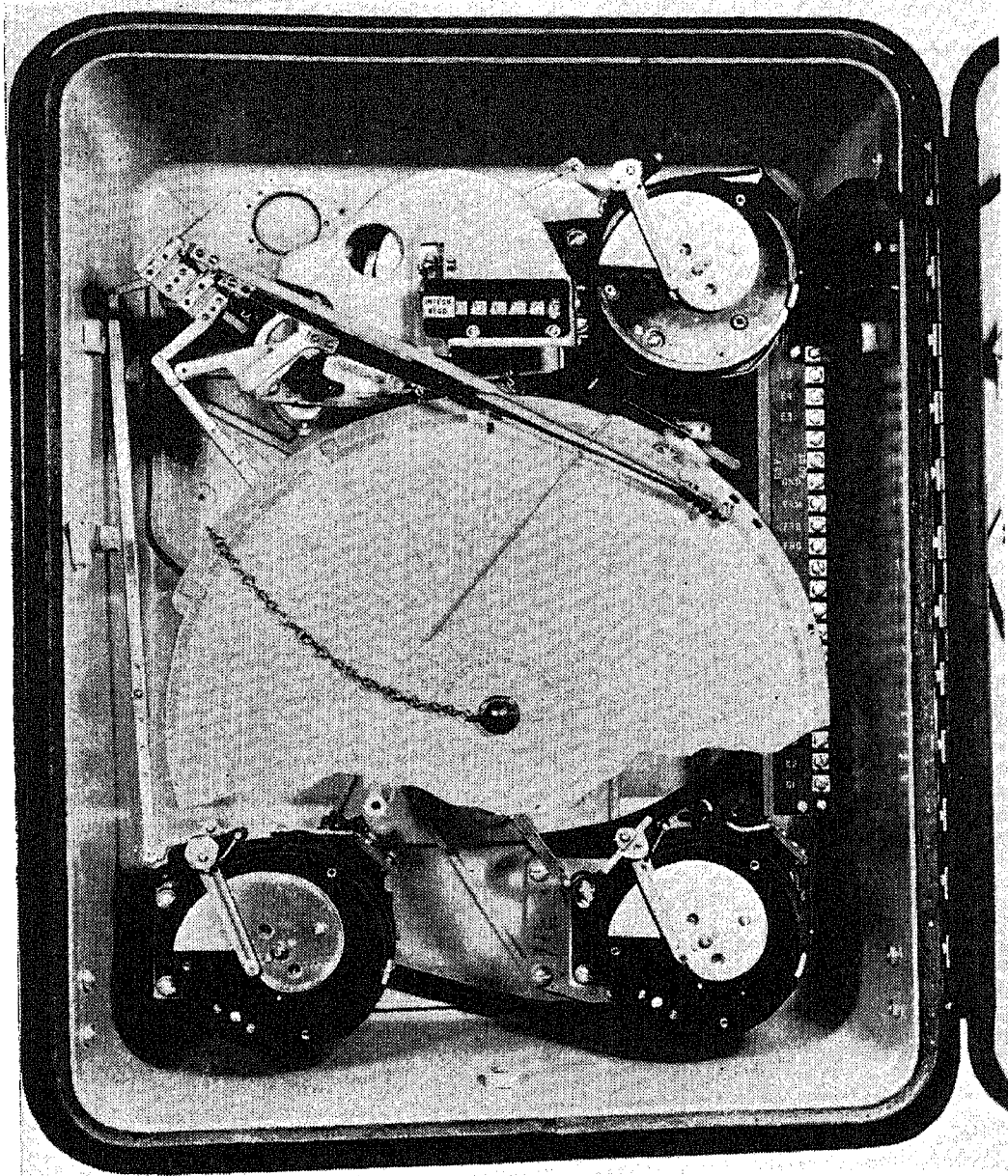


Figure 12 (left). A 3-element receiver for recording pressure, temperature, and compensated flow rate with integration of flow

Figure 13 (above). Nonrecording receiver for computing, controlling, and retransmitting

the entire metering could be accomplished by a single, multifunction type of recording receiver.

Figure 12 shows the 3-pen recording receiver presented diagrammatically in Figure 11. The motor-driven slide-wire receivers for temperature, pressure, and compensated flow drive color-coded recording pens to obtain continuous, identifiable records on the chart. The flow integrator is of the mechanical type and permits determination of total flow over a specified period of time. The servoamplifiers for the balancing circuits are mounted on the back of the instrument cases and are connected into their various circuits by means of plug and socket connectors for ease of servicing.

### Control

In most applications the ultimate objective of measurement is automatic control of the process. There is increased acceptance in the process industries of the basic electronically balanced null recorder and its many variations as an analogue computer to obtain derived functions which are more significant to the process output than the primary variables of pressure, temperature, level, flow, and so forth. The addition of auto-

matic control to these telemeter computers is a further step in the complete mechanization of the process plant.

A newly developed nonrecording type of computing receiver, incorporating in a relatively small instrument case retransmitting elements for either pneumatic or electric control, is shown by Figure 13. The use of the double cam balancing unit for the transformer follow-up and the addition of two slide-wires for resistance functions makes the application of this unit relatively universal. The balancing amplifier has been made as small as possible by the use of miniature tubes and functionally designed and selected components. It is mounted inside the case, but is easily removed for inspection and replacement. The motor-operated cam and slide-wire unit, the adjustable core transformer, and the pneumatic pilot are also subassemblies which can be removed readily and replaced. This type of receiver will be used in applications where the available space is severely limited and where recording of the individual functions is not required. The addition of control to the more conventional recording receivers, such as shown in Figure 12, is common.

It should be obvious that time and space limitations prevent the description

of the more elaborate and complete computing and control systems which would be used to meter and control a complex process. The number of combinations of metering and control which can be incorporated even in one case, such as described in Figure 12, is quite large. Theoretically, there appears to be practically no limit to what can be done with these combinations of relatively standard units and circuits. Unquestionably, there is an increased tendency for automatic supervision by remote metering and control of various process functions, ranging all the way from the manufacture of cement to the generation of steam. The automatic computer permits further mechanization of these processes. For example, the generation of steam in the large central stations is accomplished almost completely automatically to achieve high efficiency. In the smaller power plants and in other processes, the desirability of complete automatic supervision depends on factors such as cost, reliability, safety, and the degree to which the special techniques and arts required in the process can be reduced to a science.

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## No Discussion

# A Long-Distance Multipoint Telemetering System Using Teletype Transmission

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**Synopsis:** This paper describes a point-to-point telemetering system using pulse width modulation and sequential scanning for transmission of several measured variables over a single circuit by teletype. The system, which is essentially electromechanical rather than electronic, operates on the basic principle whereby the magnitude of a measured variable is converted into a pulse of proportional time width. The system was developed specifically for remote monitoring of pumping stations on a pipeline to permit automatic operation from a central dispatch office. In the design of the equipment emphasis was placed on requirements of reliability, sturdiness, simplicity, and ease of servicing rather than on factors such as high speed and excessive sensitivity.

## Electric Telemetering in Process Industries

**E**LECTRIC telemetering has been used in the electrical industry for many years as a means of operating power systems, generating stations, and substations.<sup>1</sup> The application of electric telemetering to the field of industrial processes, while relatively more recent, is becoming increasingly important. Processes which depend on telemetering for operation include the chemical industries, the making of iron and steel, the transmission and distribution of natural gas, the refining and transmission of oil and oil products, the operation of atomic reactors, and many others.

Short-range or local telemetering, frequently called remote metering, is commonly used inside large plants. Typical systems employ potentiometric and bridge balance circuits and electronically operated instruments as receivers. Air-operated telemetering for these applications offers strong competition to electric

systems up to distances of several thousand feet, particularly for the measurement and control of the nonelectrical variables such as pressure, temperature, level, and flow. The electric systems have the advantages of greater range and almost instantaneous response, but since the accuracy and resolution is affected by the attenuation of the transmitted signal, the longest distances normally used are approximately 5 miles for a-c systems and up to 50 miles for d-c potentiometric systems. In applications where the system involves distances of the order of 100 or even 1,000 miles as, for example, an oil pipeline system, measured quantities are converted into some factor of time such as the frequency of a periodic signal or the time duration of a pulse.

The system described in this paper, while utilizing some of the devices of local telemetering, is of the long-distance, time-factor type. Its operation depends on a number of principles which individually are not new in the present art of telemetering<sup>2</sup> but in combination result in a system of considerable flexibility which is capable of telemetering measured quantities over commercial telephone or telegraph circuits or by radio link any desired distance.

The design of this system is basically of an electromechanical nature utilizing a minimum number of electronic components. Electronic amplifiers and relays actually are used only in the local receivers and in the "sweep balance" detectors, since the requirements of the specific application as to speed and resolution did not warrant an all-electronic system. However, it should be apparent that the general principle of operation of

the system is not restricted to the specific design which was developed for the pipeline application.

This long-distance telemetering system was developed as part of a co-operative project to provide completely automatic operation for pumping stations on a products pipeline of the Shell Oil Company running from Wood River, Ill., to Toledo and Columbus, Ohio. These pumping stations are controlled from the central dispatching office located in New York City.

The terminology employed in this paper is taken both from the point-to-point and the mobile or radio telemetering fields and the choice of a term used in describing a principal function is selected on the basis of its accuracy and conciseness. It is important to note that basic principles of operation are common to both groups of telemetering and the differences in equipment exist principally because of the special techniques that are utilized in the two general systems. These differences in techniques depend primarily on the differences in requirements which exist between these two general fields of application. From the technical standpoint it is believed that mobile telemetering to a considerable extent has become the leader principally because of its utilization of the high speed of response, compactness of design, and mobility of the radio link. Consequently, it is expected that mobile telemetering will influence the design of point-to-point or ground systems to a considerable extent in the future. Furthermore, it is probable that in systems developed for telem-

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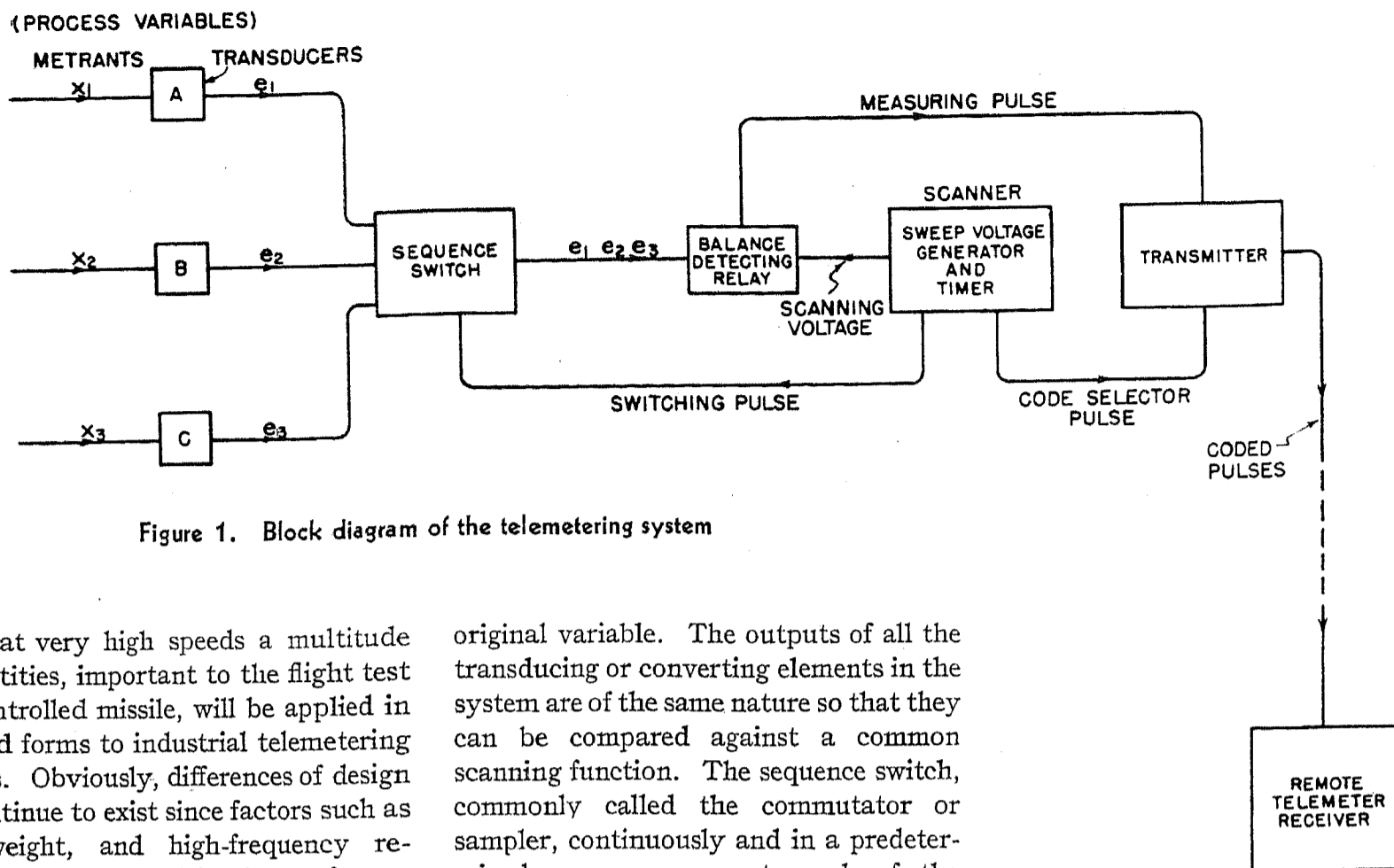


Figure 1. Block diagram of the telemetering system

etering at very high speeds a multitude of quantities, important to the flight test of a controlled missile, will be applied in modified forms to industrial telemetering systems. Obviously, differences of design will continue to exist since factors such as size, weight, and high-frequency response are not critical in the stationary systems, whereas long life, reliability, ease of service by nonelectronic personnel, and other factors are important in the industrial field.

## The Basic System

The telemetering system described in this paper is essentially a combination of local telemetering units of the servo-actuated position type<sup>3</sup> coupled to a long-distance system which operates on a general principle of what the mobile telemetering engineers call a "time division, pulse coded multiplexing system." Actually, it is also what the point-to-point telemetering engineers designate as an impulse duration multiplex or multipoint system. In this scheme each measurand (measured quantity) is sampled in a sequence by means of a scanner and then transmitted to the final receiving station or stations as a series of coded characters. Each channel of information is identified by the code selected and each measurement of the various quantities is sampled in sequence and transmitted as a complete series of coded characters by converting the width of the pulse as determined by the scanner into a proportional number of coded characters. Figure 1 is the basic diagram of the system. The measurands may be electrical but in process industries are more frequently non-electrical quantities such as pressure, flow, level, and so forth. Each of the quantities is converted either directly or through several stages of local telemetering into an electrical quantity proportional to the

original variable. The outputs of all the transducing or converting elements in the system are of the same nature so that they can be compared against a common scanning function. The sequence switch, commonly called the commutator or sampler, continuously and in a predetermined sequence connects each of the transducers into the scanning circuit where a comparison is made between their outputs and the time varying output voltage of the scanner. If the output of the pickup is a voltage directly proportional to the measurand, the output of the scanner during the rise period is made a linear function of time. As will be explained in detail later, the function of the scanner is to convert the magnitude of the output of each transducer into a proportional time pulse.

The proportional time interval is represented by the time of dwell of an electric contact in this system but also may be represented by the period of conduction of an electron tube. This pulse may be transmitted without further modification over a wire link. If a radio link is used, an amplitude- or frequency-modulated carrier is required.

The transmitted quantity may be identified at the receiver by its chronological position in a train of impulses such as might be indicated on a cathode-ray oscillograph. In the specific system, developed for the pipeline telemetering, identification is accomplished by converting each pulse into a group of coded teletype characters. The coding is accomplished by synchronizing the teletype equipment with the sequence switch.

## Description of the Function of the Elements in the System

### PICKUPS

There are many primary detectors or pickups in use today whose purpose is to

convert the measurand to some electrical quantity which can be measured conveniently. Figure 2 shows a number of transducers, which convert the process variables such as pressure and flow proportional alternating voltages means of differential transformers, whose output voltages are proportional to the core position.<sup>3</sup>

The pickups shown by this figure are employed where the basic measuring elements are mechanical devices such as bellows, Bourdon tubes, floats, and so forth. This type of mechanical link produces a position or motion proportional to the measurand which then is converted into an electrical variable by means of an adjustable core transformer coupled to the mechanical device. It will be noted that these systems represent two or sometimes more stages for the conversion of the measurand into an electrical variable. On the other hand pickups such as thermocouples and resistance thermometers are primary detectors which perform the conversion in a single stage.

Figures 2(A), (B), and (C) show pickups whereby pressure, differential pressure, and flow respectively may be measured and converted to proportional voltages. For variables such as pressure level the arrangement shown in Figure 2(D) is suitable if a record is desired at the transmitter. In the case of nonlinear variables a secondary pickup is employed as indicated in Figure 2(E). Here the primary detector is balanced by a mo-

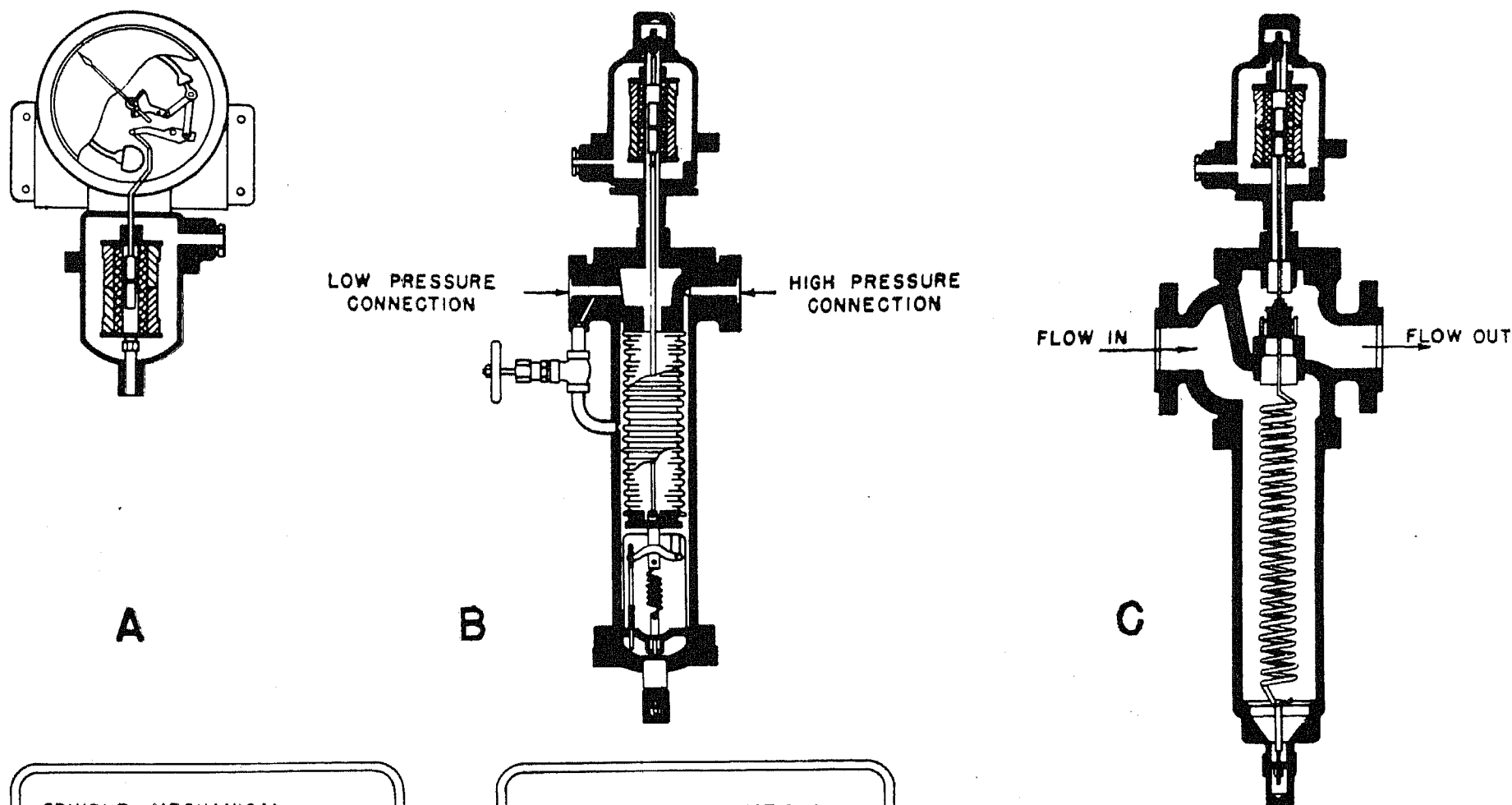


Figure 2 (above). Pickups of the differential transformer type

(A) Pressure. (B) Differential. (C) Flow. (D) Internal mounting in mechanical meter. (E) Internal mounting in electronic-type recorder

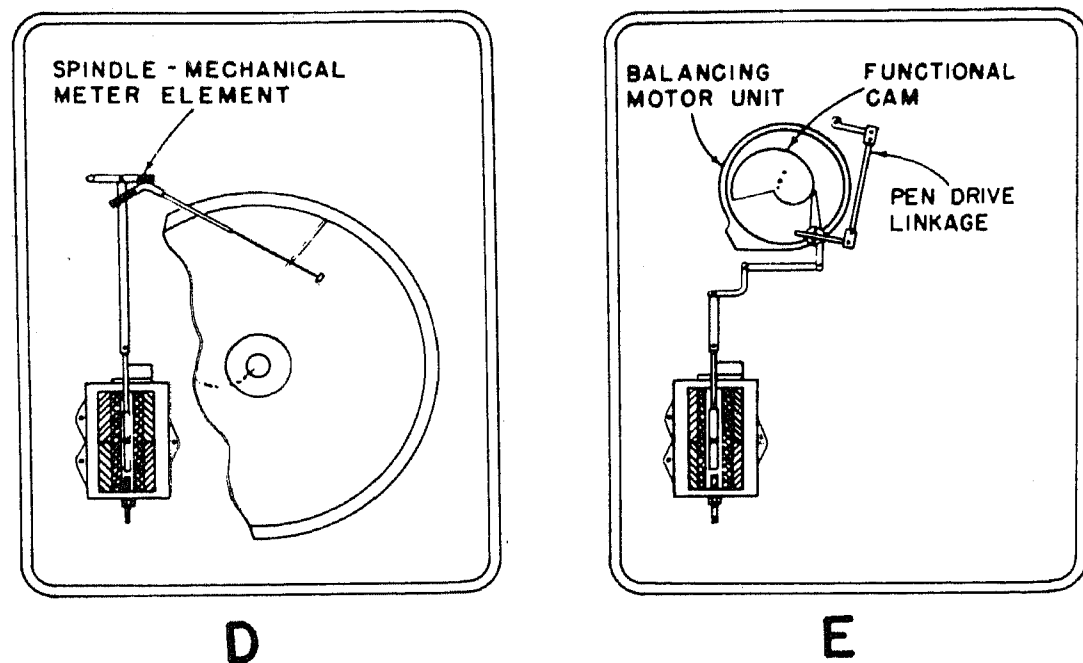


Figure 3 (below). Pickups of the resistance type

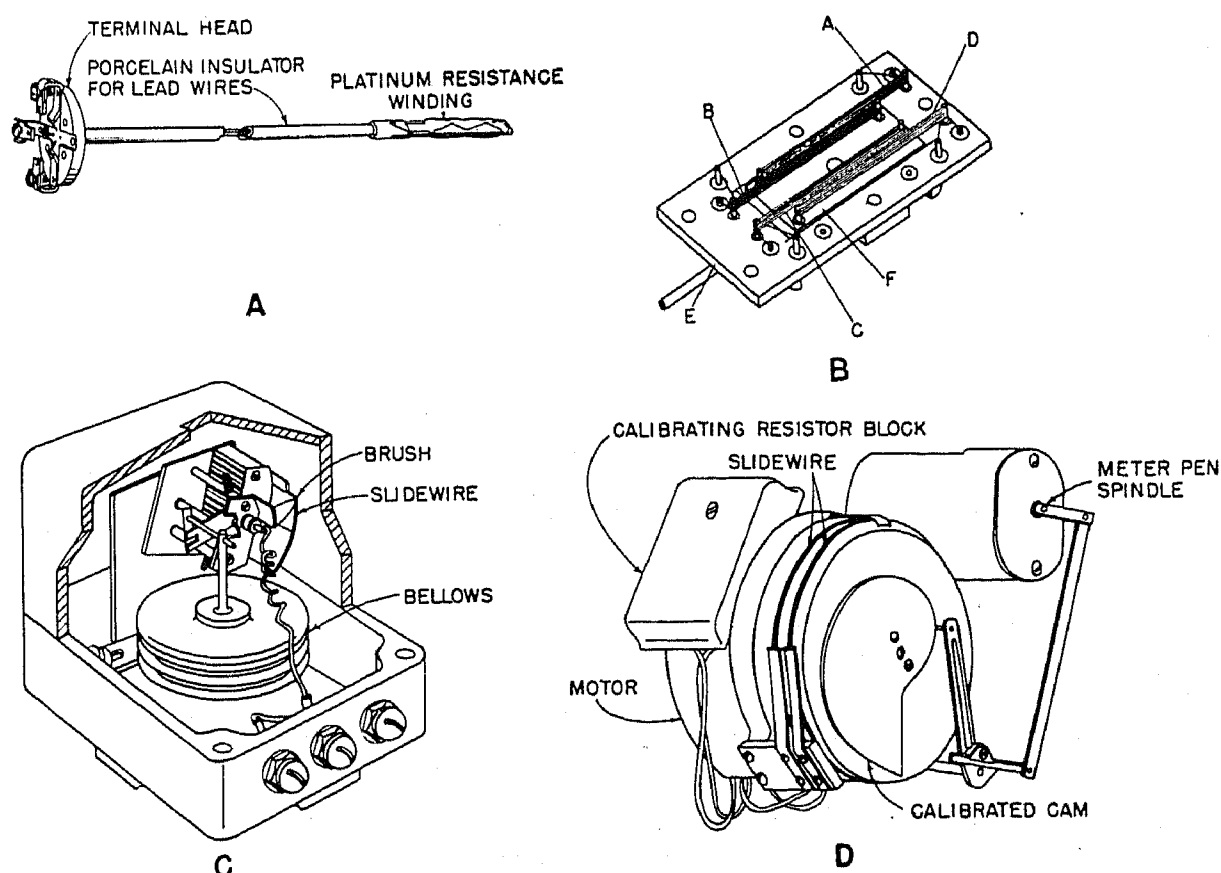
(A) Temperature pickup, resistance thermometers. (B) Strain gauge pickup, Stratham unbonded type. (C) Pressure pickup, Gianinni type. (D) Motor-driven slide-wire, balancer, and retransmitter

driven slide-wire unit and retransmitted to the differential transformer by means of the correct nonlinear function cam to give an output voltage which varies linearly with the measurand.

Figure 3 shows several pickups of the resistance type. The resistance thermometer and the strain gauge are primary detectors in which the resistance varies directly with the measurand. The pressure pickup shown in Figure 3(C) functions by positioning the slider of a low-torque potentiometer. Figure 3(D) shows the manner in which a retransmitting slide-wire is actuated by the motor of a self-balancing recorder.

#### THE SCANNER

The scanner shown in Figure 4, consists of a synchronous motor driving a linear rise cam which actuates the core of the differential transformer in such a way that its output voltage varies pro-



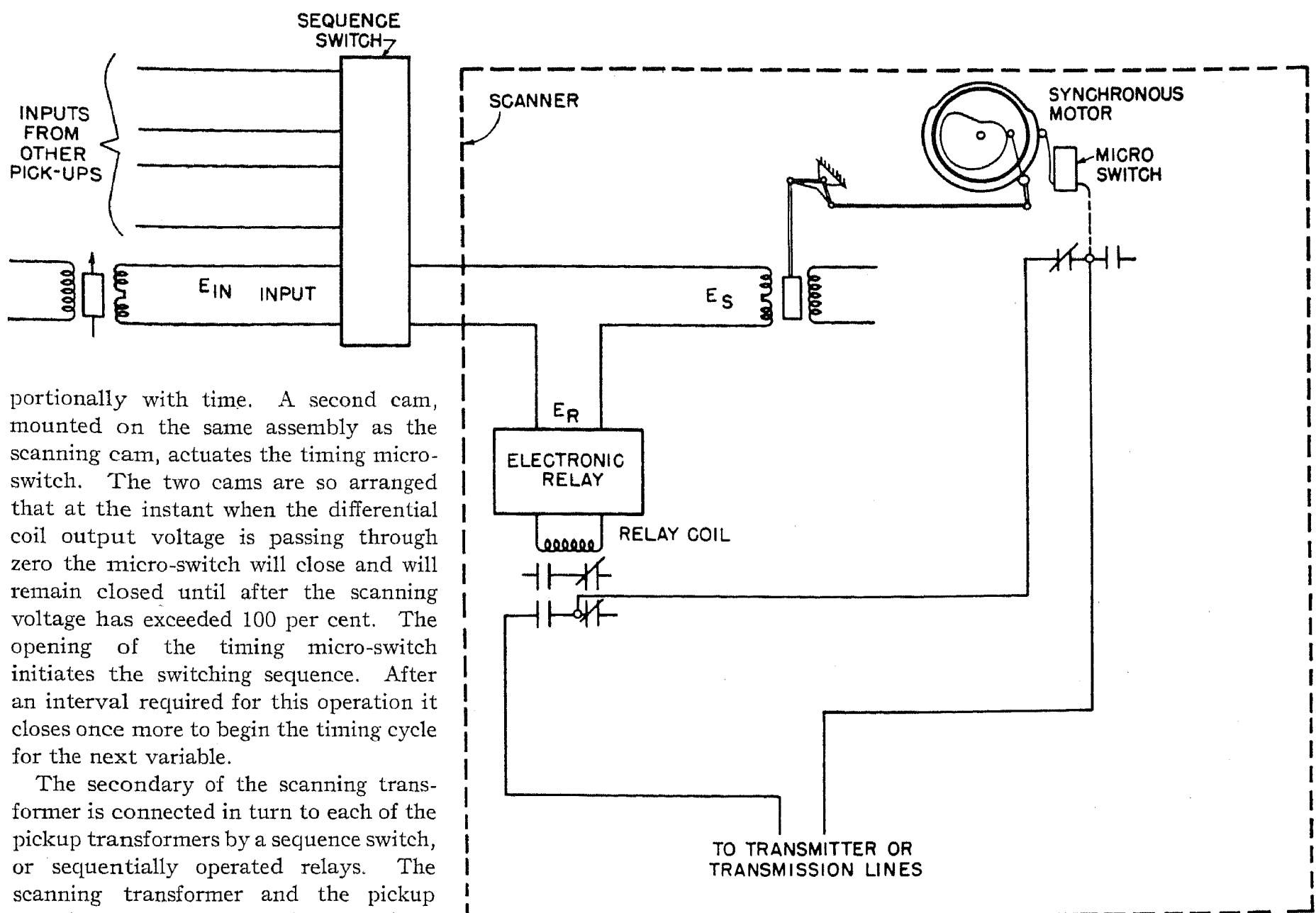


Figure 4. Elementary diagram of the scanner

portionally with time. A second cam, mounted on the same assembly as the scanning cam, actuates the timing micro-switch. The two cams are so arranged that at the instant when the differential coil output voltage is passing through zero the micro-switch will close and will remain closed until after the scanning voltage has exceeded 100 per cent. The opening of the timing micro-switch initiates the switching sequence. After an interval required for this operation it closes once more to begin the timing cycle for the next variable.

The secondary of the scanning transformer is connected in turn to each of the pickup transformers by a sequence switch, or sequentially operated relays. The scanning transformer and the pickup transformer are connected to obtain a voltage proportional to the difference of their outputs. This difference or error voltage is applied to the electronic relay. This relay is so arranged that when the input voltage to it is of one phase its output relay coil is de-energized, and at the instant the phase changes the relay is energized, closing its contacts.

The scanner operates in the following manner. Starting at time equal to zero, when the timing micro-switch closes, the scanning voltage is zero and begins to increase linearly with time. The voltage to the electronic relay is a maximum at  $T=0$  since

$$E_r = E_{in} - E_s \text{ and at } T=0, E_s=0$$

This relay voltage will decrease linearly with time as the scanning coil voltage increases and approaches the value of the output voltage of the pickup. At the instant when the two voltages are equal, the electromagnetic relay on the output of the electronic unit will be de-energized thus opening the circuit to the transmitter or transmission line and completing the period of transmission of intelligence.

The curves of Figure 5 indicate diagrammatically the operation of the electronic relay. Three curves of relay input voltage (for three different signal inputs) on a time base are shown in Figure 5(B)

showing how the magnitude of the signal voltage determines the pulse length. Figure 5(C) depicts one cycle of the scanning voltage.

Over the measurement interval the reference voltage of the scanner is given by the equation

$$e_s/E_s = t/T_0 \quad (1)$$

where

$e_s$  = the voltage output of scanner at time  $t$   
 $E_s$  = the voltage output of the scanner at time  $T_0$ , the end of the scanning period

The output of the linear transducer is given by

$$e_x/E_x = x/X_0 \quad (2)$$

where

$e_x$  = voltage output of transducer at a value of variable  $x$   
 $E_x$  = voltage output of transducer at a value of variable  $X_0$  (100 per cent)

The duration of the measurement contact is obtained by equating  $e_s$  to  $e_x$  and solving for  $t$ . Hence

$$t = (E_x/E_s)(x/X_0)T_0 \quad (3)$$

$$\text{If } E_s = E_x$$

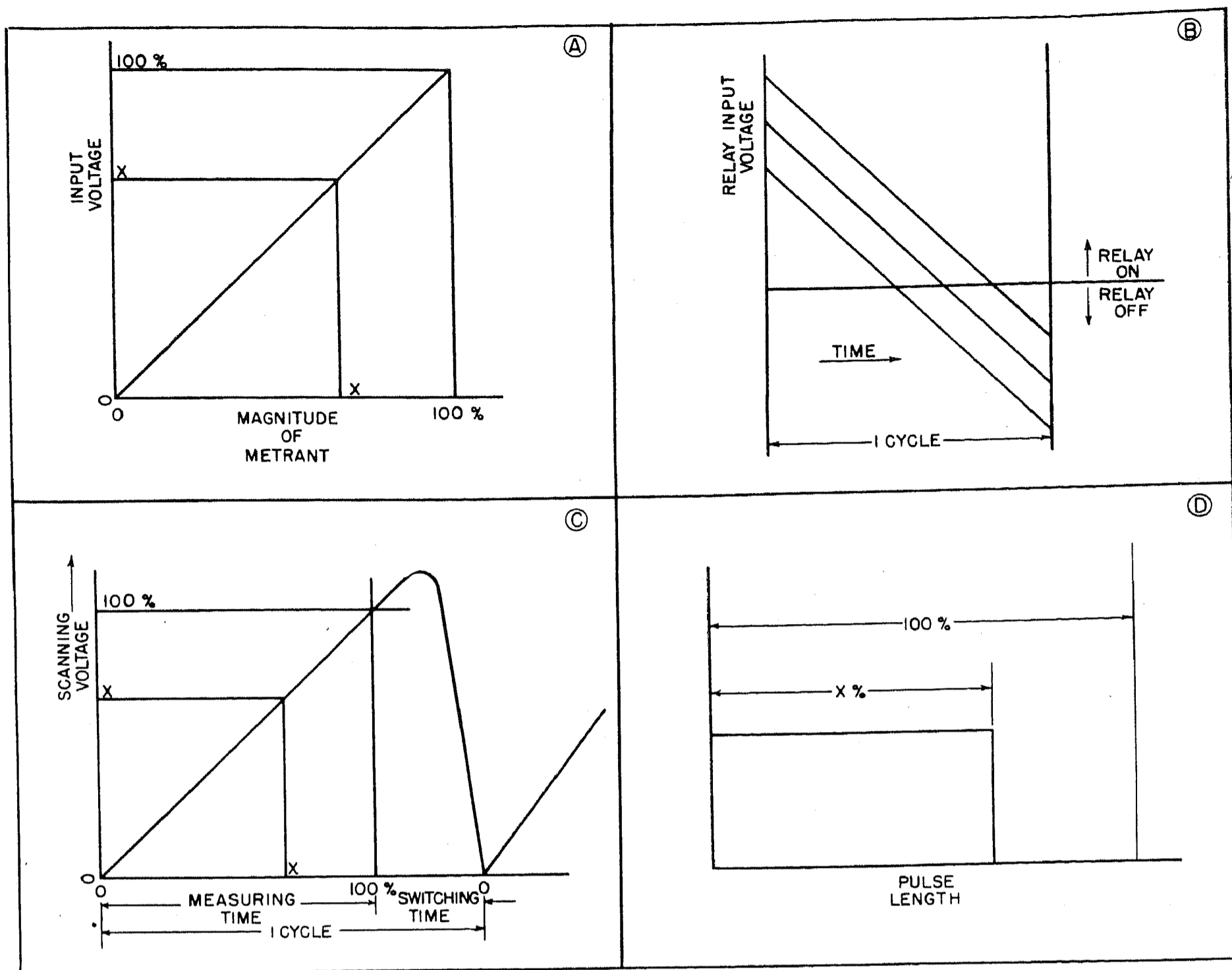
$$t = (x/X_0)T_0 \quad (4)$$

Equation 4 shows that the duration of the measurement contact is directly proportional to the measurand  $x$ .

If the output of the transducer is a nonlinear function of the measurand, this function can be extracted by using a servo-actuated receiver to operate through a cam, the core of a retransmitting transformer. For example, the voltage output of the differential transmitter shown by Figure 2(B) is proportional to the square of flow. In this case the retransmitter cam has a square root rise characteristic so that the output of the retransmitter transformer will be proportional to flow. If, however, all of the quantities to be scanned have the same nonlinear characteristic, the cam of the scanning unit may be shaped so that its rise is a similar nonlinear function of motor shaft rotation. For example, in the case of the differential-type flow meter

$$e_x = E_x(x/X_0)^2 \quad (5)$$

If the rise of the scanning cam is made proportional to the square of the angle



of rotation of the scanning motor shaft

$$e_s = E_s(t/T_0)^2 \quad (6)$$

Consequently, if  $E_s = E_x$ , the electronic relay will be energized for the period  $t$  obtained by equating 5 and 6.

$$t = (x/X_0)T_0 \quad (7)$$

as in the linear system.

Figure 5 (above). Graphs of process magnitude, voltage-time relationships

Figure 6 (below). Map indicating locations of pipeline pumping stations

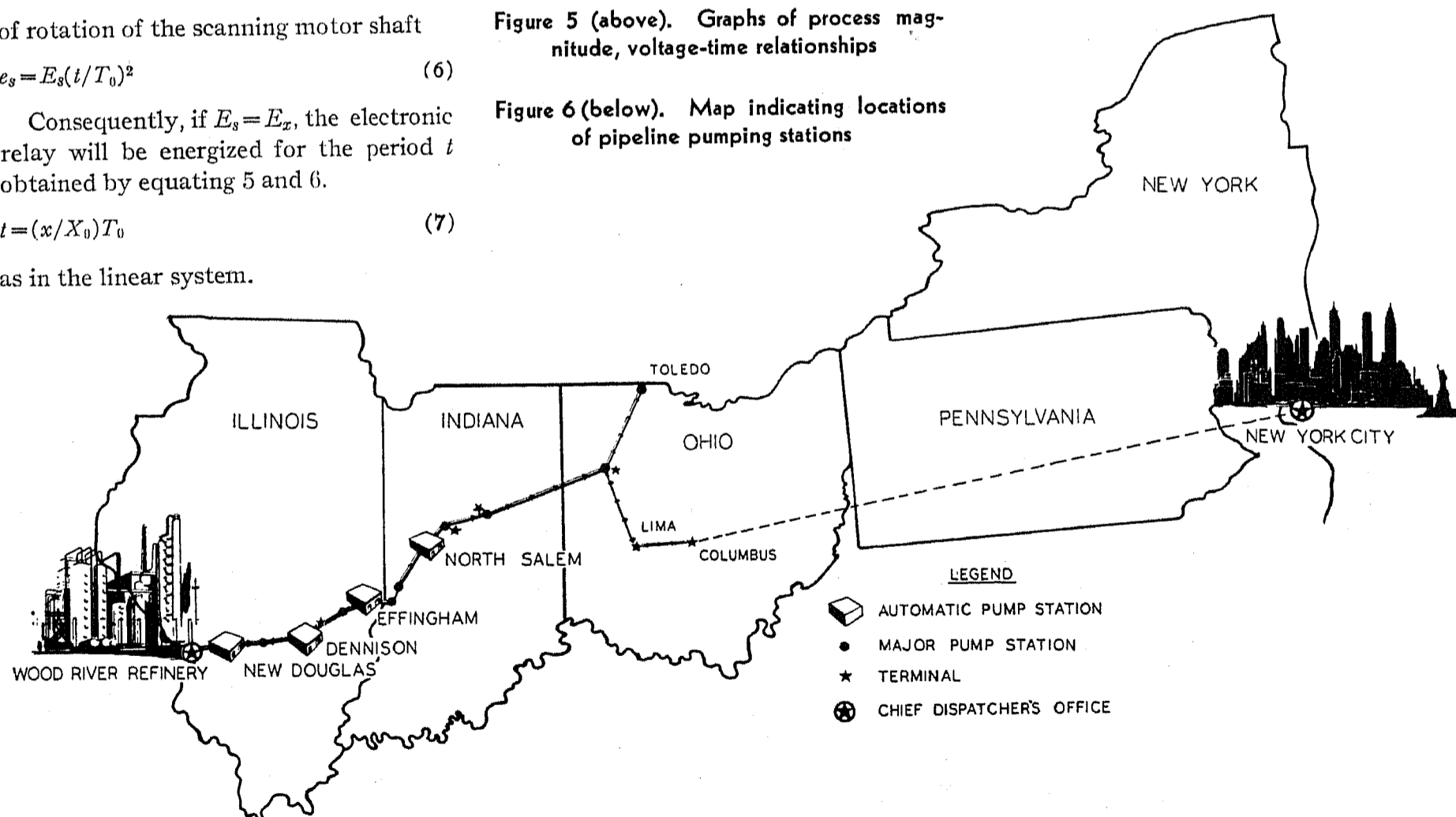
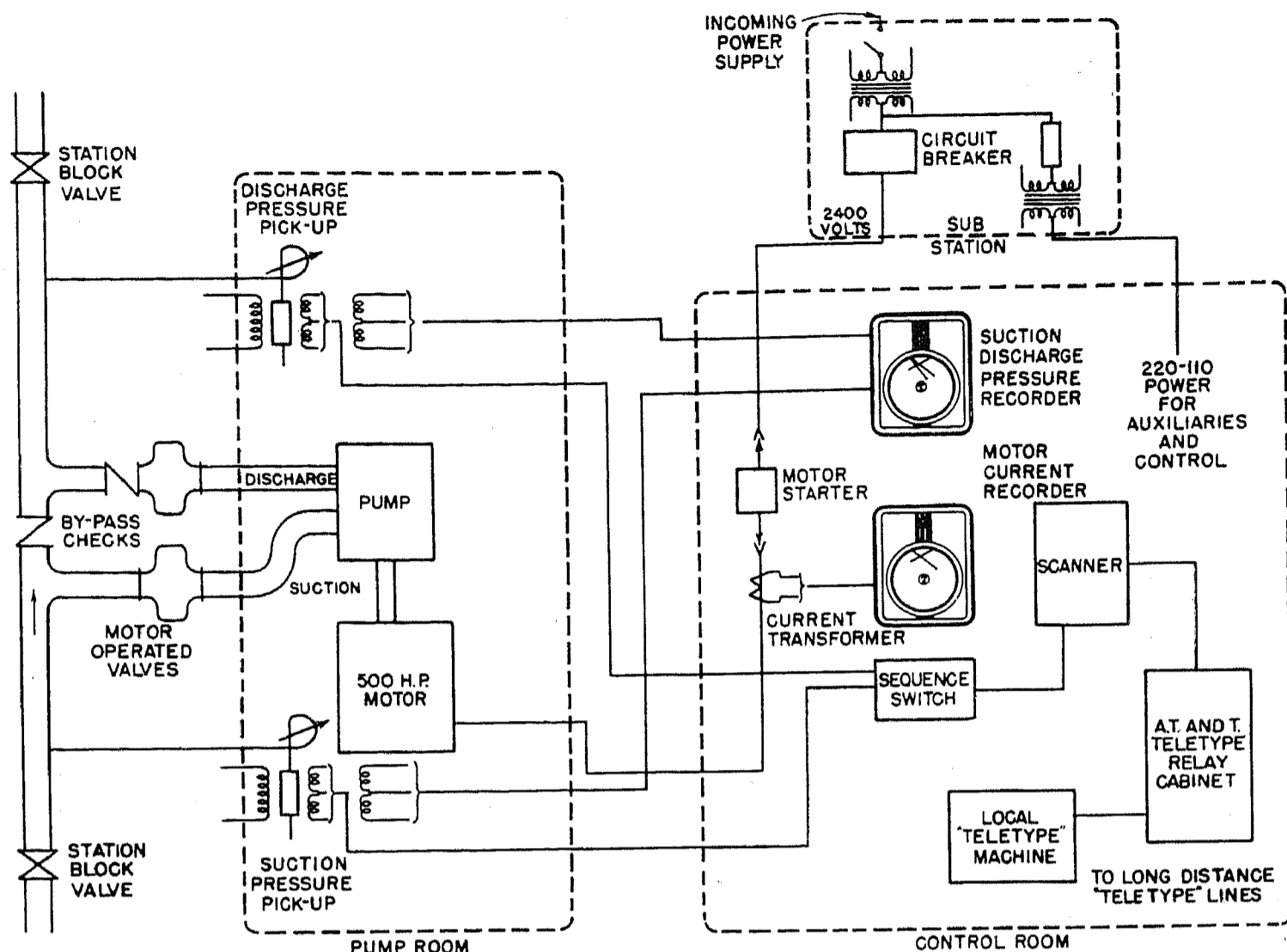


Figure 7. Simplified block diagram of station arrangement



It is entirely feasible to use a rotating slide-wire as the scanner device, and for high-speed telemetering a calibrated electronic generator could be used. For the requirements of the pipeline telemetering system the cam-operated scanner has a number of basic advantages such as elimination of sliding contacts, ruggedness, and ease of calibration.

### Transmission Link

One method by which the intelligence may be transmitted and coded is by the use of the teletype system. While a detailed explanation of the various circuits required to achieve this is beyond the scope of the paper, the general operation of the system is given in the following paragraphs.

An electric contact whose duration is proportional to the measurand is supplied by the electronic balance detecting relay of the scanner to the teletype transmitting equipment. The teletype apparatus sets up coding letters for each variable, such as is shown next, using the letter *S* for suction pressure, *T* for temperature, *F* for flow, *L* for level, and so forth. The equipment starts printing at a constant rate the appropriate letter at the beginning of the scanner timing cycle and stops when the scanner contacts open. Thus, the number of letters in any one sequence is proportional to the variable. At the

end of the transmission of intelligence, a code letter such as *PQ* is sent to indicate the location of the transmitter in cases where there may be similar intelligence channels from other remote stations.

SSSSSSSSSSSSSSSSSSSS *PQ*  
 TTTTTTTTTTTTTTTTTTTT *PQ*  
 FFFFFFFFFFFFFFFF *PQ*  
 LLLLLLLLLLLLLLLLLLLLLL-  
 LLLLLLLLLLLLLL *PQ*

The teletype channel has the advantage of using comparatively standard equipment and easily furnishing the intelligence required both at the transmitter and receiver and all other teletype sets connected to the particular network. It has the disadvantage that the resolution is limited by the speed at which the teletype may be operated. In most cases, however, this is not a serious drawback.

### Pipeline Application

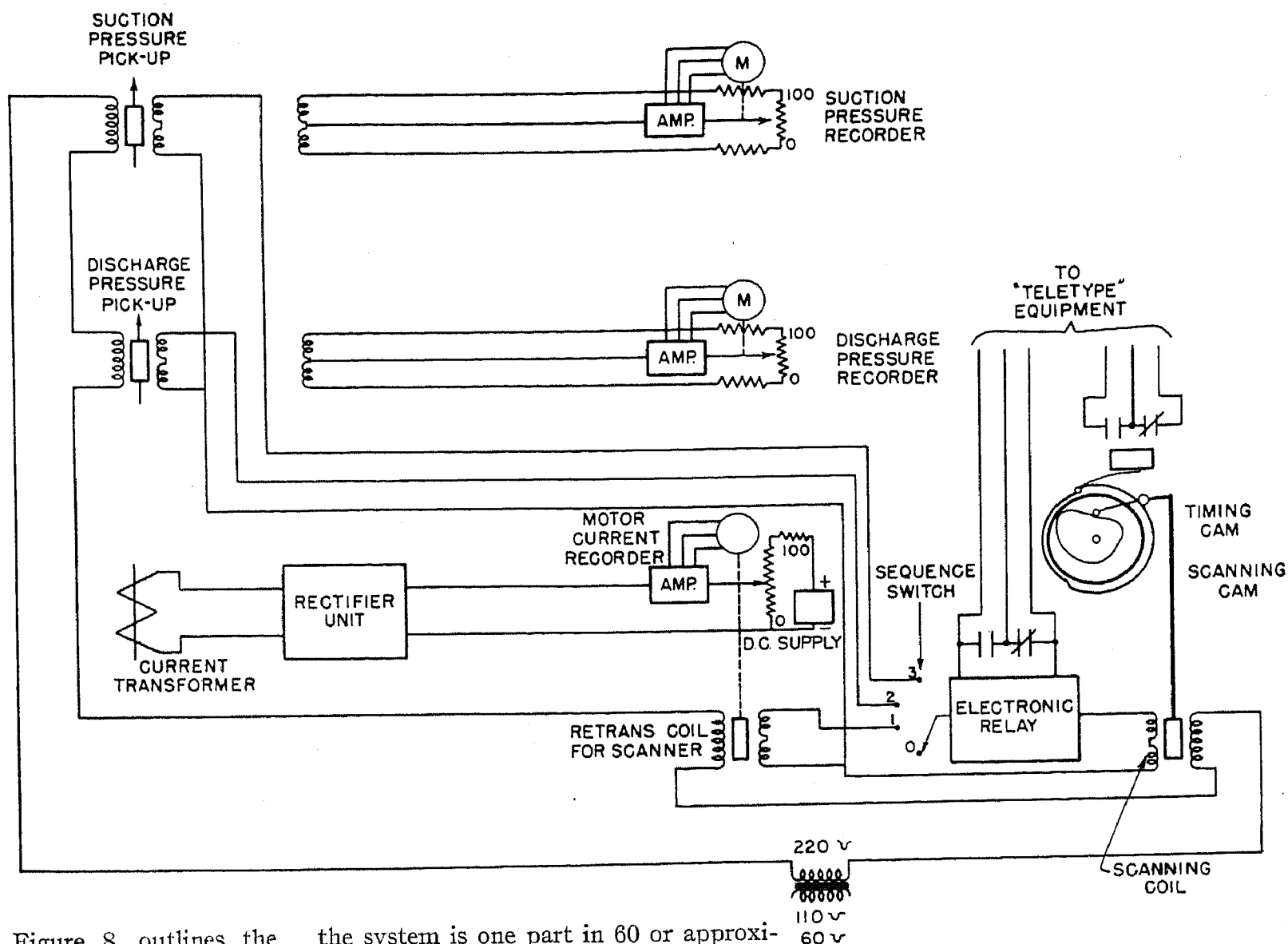
This system was first used on automatic pumping stations for a pipeline of the Shell Oil Company. It is installed, as shown in Figure 6, on the four automatic booster stations located at New Douglas, Dennison, and Effingham, Ill., and North Salem, Ind. The stations make a much higher throughput possible than usual in an 8-inch line as well as increasing overall operating efficiency.

The operation of the four stations is

entirely controlled from the central dispatch center in New York City. Various products are pumped in turn through the line and it has been found that the slight intermixing produces no harmful results in the quality of any of the products. The location of these products is traced on a control board which simulates the pipeline. Thus, the dispatcher in New York has a completely integrated picture of the location and quantity of all products in the whole pipeline as well as the pressures and pump motor current of each of the four booster stations. Since the dispatcher has control of these stations, as well as complete operating information, he is in a position to operate the line in a more efficient manner.

The telemetering equipment, as previously described, furnished information regarding suction pressure, discharge pressure, and pump motor current. A block diagram, Figure 7, indicates the telemetering equipment supplied for each of the four booster stations. The suction and discharge pressures are measured by gauges installed in the pump house and locally telemetered to the control room where they are recorded. A motor current recorder also is located on the same panel. Electrically isolated voltages from the three transmitting differential transformers are connected to the sequence relay and in turn to the scanner for long distance telemetering. A simplified sche-

Figure 8. Simplified schematic diagram of telemetering system



matic diagram, Figure 8, outlines the circuits used in the measuring and telemetering equipment supplied by Bailey Meter Company.

To obtain readings of telemetered variables, the dispatcher at New York City dials a code number which operates the equipment at any one of the four chosen stations and automatically telemeters the pressures and pump motor current in the form as shown previously. At the end of one telemetering cycle of the three variables, the equipment automatically shuts off and the dispatcher, if he desires, then may obtain similar information from the three other stations. The dispatcher also may start and stop the main pump motor, lubricating oil pump, suction and discharge valves, and other necessary equipment. In an emergency, he also may sound a siren to summon the operator who is located at each station for maintenance and emergency purposes.

All the information received in the New York dispatcher's office is simultaneously printed on all the other teletype machines which are connected to the same system. The teletype machine is operated at a speed of 368 characters per minute or 163 milliseconds per character. The total cycle period for the scanning of one measurand is 15 seconds, or 45 seconds for a complete intelligence report from any one station. The 100-per-cent measurement time was selected at 60 characters or 9.783 seconds. Thus, the resolution of

the system is one part in 60 or approximately 1.7 per cent. The remaining portion of the cycle, 5.2 seconds, is allowed for the various switching operations involved to transfer from one pickup to the next, change code letters, and so forth. If a lower resolution were permissible, the cycle time could be reduced proportionally, and similarly for a higher resolution, a longer cycle would be employed.

### Operating Experience

This system has been in operation since September 1950. Some trouble was experienced at the beginning with servo-amplifier failure, caused by faulty components, but this was easily remedied since these units are standard plug-in assemblies. Since then the equipment has been found to be completely reliable due to its inherently simple nature, lack of delicate components, and over-all ruggedness. It serves its purpose admirably in integrating all information and supervisory control at one centralized point and thus permitting much more efficient utilization and operation of the pipeline system.

### Conclusions

It will perhaps appear obvious to the electronic engineer in the field of mobile telemetering that the utilization of electronic techniques could have been made

in any or all of the elements of the system described. For example, an electronic scanning system could have been used in place of the electromechanical system, resulting in much greater speed of scanning. Acceptance of complex electronic systems for the control and telemetering of industrial process variables has been somewhat slow, particularly where the requirements do not exclude the use of simpler electromechanical mechanisms. Where high speed and sensitivity are required and a great number of variables are to be telemetered, the electronic systems probably would be required.

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No Discussion

# Subharmonics in a Series Nonlinear Circuit as Influenced by Initial Capacitor Charge

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**S**ERIES circuits containing a capacitance and nonlinear inductance have assumed considerable importance in modern engineering design. However, knowledge of the behavior of such circuits is extremely limited because of the inherent nonlinearity of the differential equations encountered.

While considerable work has been done by the investigators of nonlinear vibrations and celestial mechanics on nonlinear differential equations in which the nonlinearity is involved in the damping term (the equation of van der Pol) or in the displacement term (Duffing's equation) or in the forcing function, very little has been done on differential equations in which the nonlinearity is involved in the inertia term. The subject is further complicated by the fact that transients in such networks may be extremely long so that solutions by a differential analyzer are lengthy and subject to cumulative errors.

Hence it is felt that at present the most reliable method of predicting with a reasonable degree of accuracy the performance of proposed circuits is from a wealth of experimental data. The data now available give considerable insight into the type of response that may be encountered. From the work of Rouelle,<sup>1</sup> McCrumm,<sup>2</sup> and others,<sup>3-5</sup> it is known that three different types of current response may be encountered in a series circuit containing a resistance, a capacitance, and a nonlinear inductance upon shock excitation. The first of these is the normal fundamental-frequency exciting current of the inductance. The second is

a much higher value of current of fundamental and higher frequencies, which has been labeled ferroresonance. The third type of response contains currents which are integer submultiples of the applied frequency. This phenomenon is called subharmonic response, and the order of this response indicates the ratio of the applied frequency to the lowest frequency component of the resulting current.

The work of these investigators gives

information as to the ranges of circuit parameters for which the different types of response may be obtained. Angello<sup>6</sup> reported the effect of the initial switching angle on the occurrence of subharmonics with constant initial capacitor charge. However, the effect of initial capacitor charge or of initial flux linkages has not been investigated by previous authors. The purpose of the investigation reported in this paper was to determine the effect of the initial capacitor charge on the occurrence of subharmonics, with the initial switching angle and initial flux linkages held constant. The investigation was restricted to third-order subharmonics as this is the order most likely to be encountered.

## Test Circuit

The circuit investigated is shown in Figure 1. Figure 1(A) shows the non-

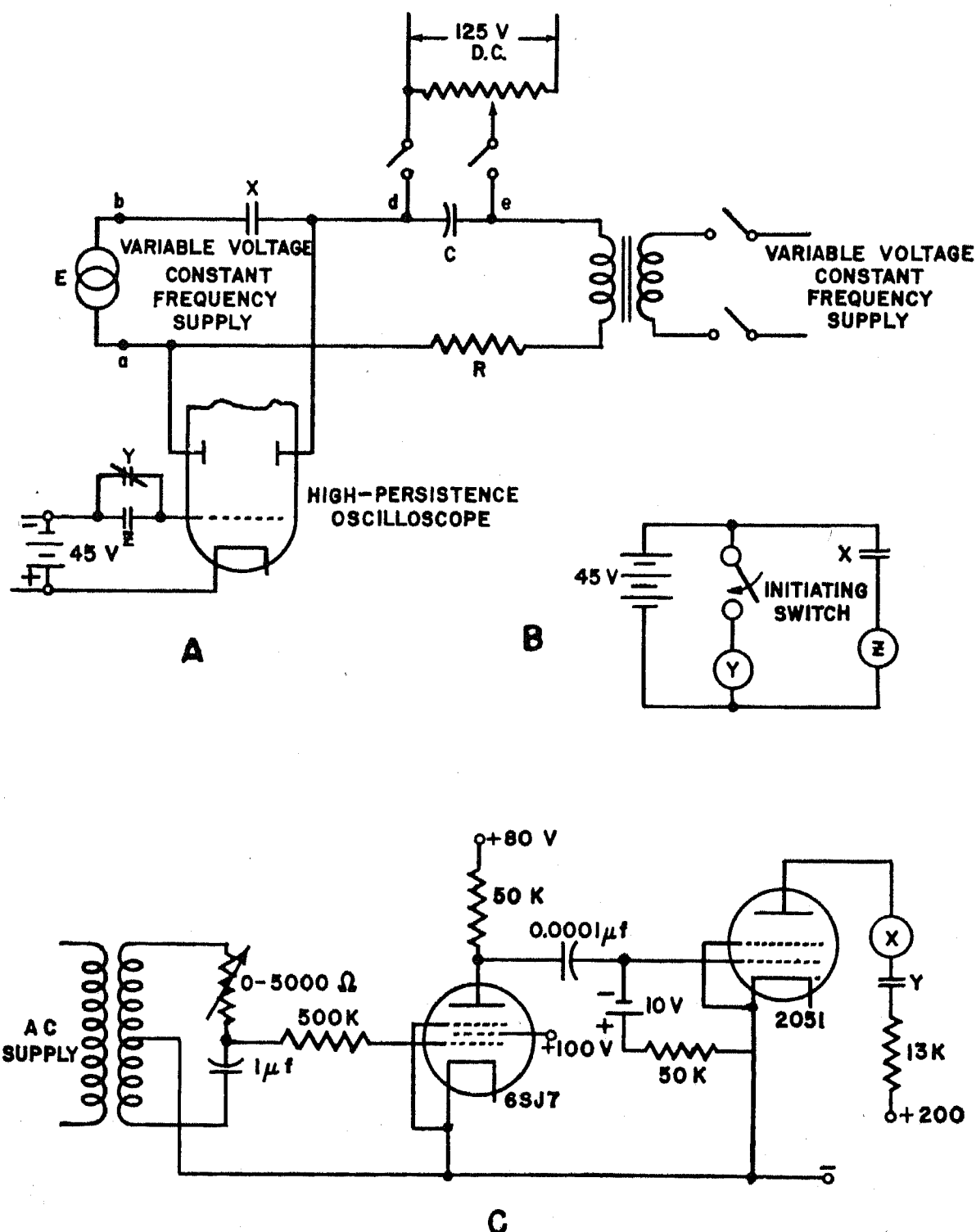


Figure 1. Test circuit

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This paper is based on a thesis submitted by Mr. Brust in partial fulfillment of the requirements for a degree of Master of Science in Electrical Engineering at The University of Texas, 1951.

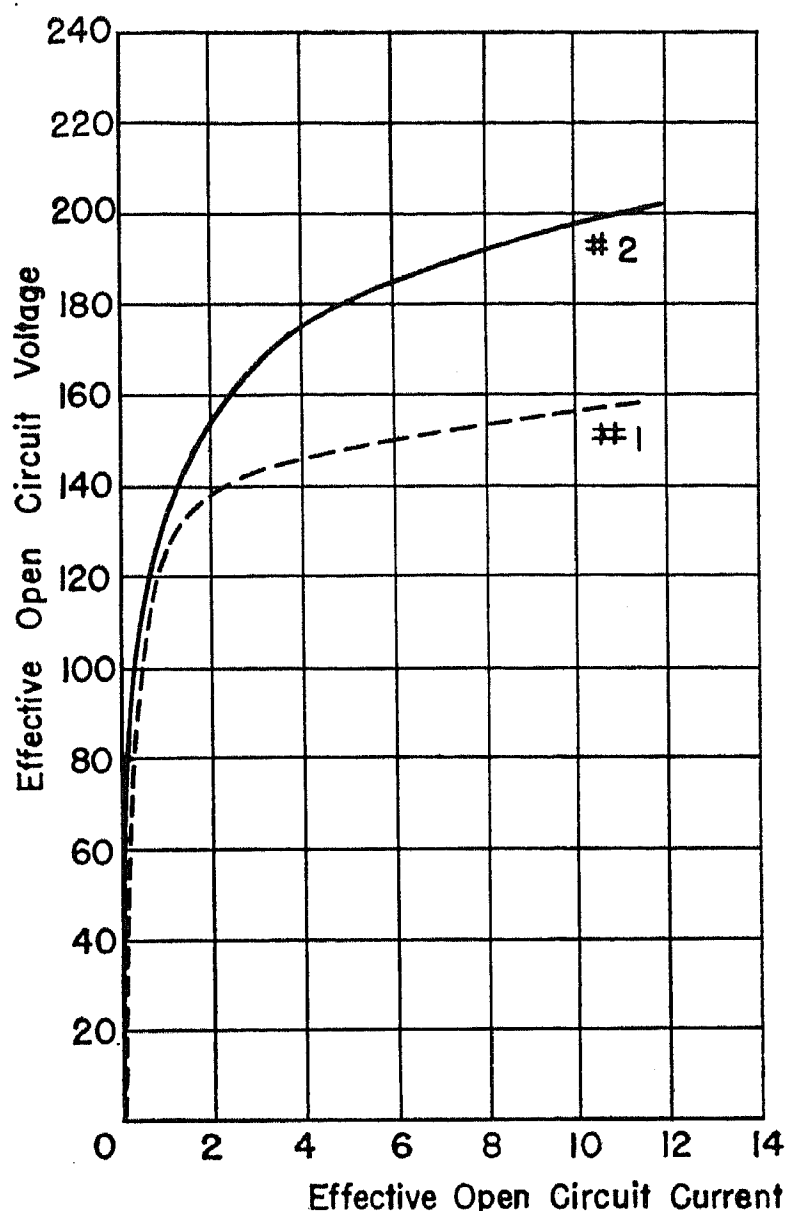


Figure 2 (left). Saturation curves of inductive elements at fundamental frequency

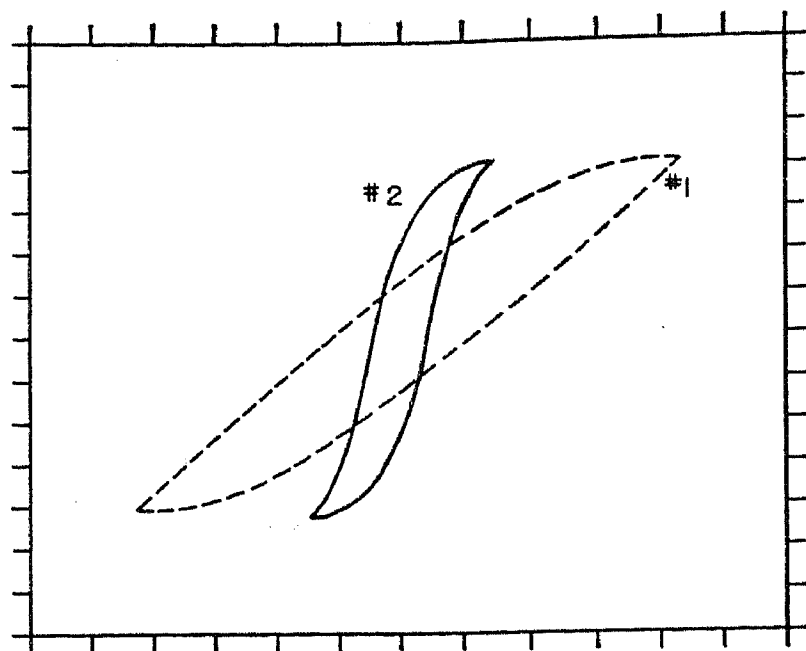


Figure 3 (right). Hysteresis loops at 57 volts rms

linear circuit containing a resistance  $R$ , a capacitance  $C$ , and a nonlinear inductance (a transformer with open secondary) connected through a relay contact  $X$  to a low-impedance variable voltage source. The capacitor was charged prior to closing the relay contact by applying a unidirectional voltage. Inasmuch as the occurrence of subharmonics was found to be influenced by initial flux linkages, the transformer was demagnetized by applying a decreasing-amplitude sinusoidal voltage to the secondary of the transformer prior to each reading.

A high-persistence oscilloscope was used to set the relay contact  $X$  to close at the desired angle and to monitor this angle continuously. To observe the angle it was necessary to blank the oscilloscope approximately one cycle after relay  $X$  operated. It was found that if the beam of the oscilloscope was left on prior to operation of the  $X$  relay the initial trace on the screen was objectionable. This was eliminated by applying a negative voltage to the grid of the cathode-ray tube, removing this bias just prior to the closing of  $X$ , and putting it back approximately 1 cycle later. Because of the high potential to ground of the grid circuit, it was necessary to isolate this circuit from the  $X$  relay.

The circuit used for control of the

switching angle is shown in Figures 1(B) and 1(C). The grid of the 6SJ7 tube was supplied through a phase-shifting network such that the phase position of the approximately square wave output of the 6SJ7 tube could be varied with respect to the applied voltage. This output was applied to the grid of the thyatron through a differentiating network with a time constant of about 5 microseconds. The phase position of the resulting pulse could thus be regulated by means of the phase shifter in the grid of the pentode. The magnitude of the pulse was controlled by adjusting the input to the phase shifter and was set to such a value that the positive pulse would raise the grid voltage of the thyatron slightly above cutoff. Thus the thyatron was triggered once every cycle with a pulse of very short duration and of variable phase position and would fire at this point if the plate circuit was closed.

#### Method of Taking Data

The data were taken as follows. The residual flux in the inductance was destroyed and the desired potential was placed on the capacitor. As soon as the initiating switch was closed, relay  $Y$  closed, which unblanked the oscilloscope and completed the plate circuit to the

thyatron. The next positive triggering impulse from the pentode fired the thyatron, energizing relay  $X$ . Relay  $X$  then energized the nonlinear circuit and picked up relay  $Z$  which again blanked the oscilloscope. The proper time interval was obtained by adjusting the spacing of the relay contacts.

Since the work of Angello<sup>6</sup> indicated that the probability of subharmonic response was greatest at zero switching angle, it was decided to energize the circuit at this point. The wave form of the current in the nonlinear circuit was observed on a low-persistence oscilloscope connected across the resistive element.

Two different inductances were used in order to determine the effect of different saturation curves. The saturation curves for the two transformers are shown in Figure 2. The hysteresis loops for the two transformers at an applied voltage of 57 volts rms, 60 cycles, are shown in Figure 3.

Three tests were made with transformer number 2: one with constant capacitance and variable applied voltage, and two with constant applied voltage and variable capacitance. With transformer number 1, one test was made with constant capacitance and variable voltage and one with variable capacitance and constant applied voltage. The values of voltage and capacitance which were held constant were selected because in a rough exploration of the complete regions of response they proved to be the values most conducive to subharmonic response.

The results obtained are shown in Figures 4 through 8 and Appendix I shows the conditions under which they were obtained.

#### Conclusions

Several facts are worthy of note. In his experimental work Rouelle<sup>1</sup> found

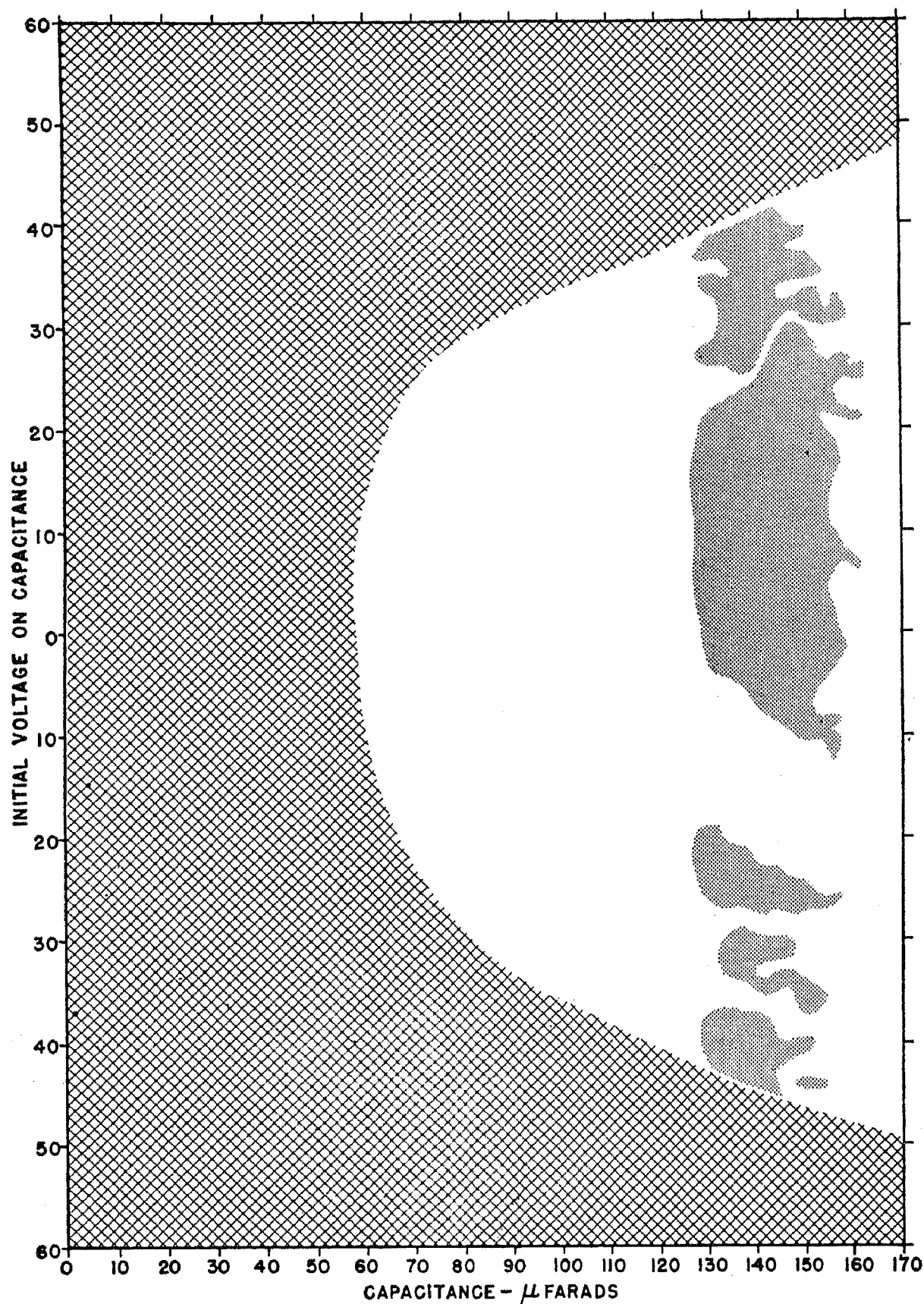


Figure 4. Areas of stable response. Transformer number 1—applied voltage, 60 volts rm

that, for random excitation, subharmonic response could not be obtained as easily for a transformer with a sharply curved saturation curve as for one in which the saturation curve had less severe curvature.

The investigation herein described supports this finding as can be seen from a comparison of the areas for the two transformers.

In his work McCrumm<sup>2</sup> found that for certain regions of capacitance, subharmonics could not be obtained for certain definite values of applied voltage although they could be obtained for values of applied voltage above or below this

value. Dehors' work<sup>7</sup> supported this conclusion. However, although a lengthy search was made on the two inductances used in the present investigation, this gap could be found on neither. One possible explanation is that transformer number 2 had a high value of conductor resistance; McCrumm pointed out that high values of resistance could destroy this "lake." And the curvature of the saturation curve for transformer number 1 may have been so sharp that this phenomenon may not have existed.

Angello<sup>6</sup> found that the initial flux linkages in the inductances had no effect on the wave forms of stable subharmonics.

This was confirmed; however, the initial flux had a definite effect on whether or not subharmonics would occur.

It should be pointed out that the response in the immediate neighborhood of ferroresonance was difficult to determine because of the magnitude of the transients encountered. In no case did a transient ferroresonant current yield a steady-state subharmonic current.

Several authors have reported that a prolonged subharmonic transient may be obtained. This was observed here, with yet another phenomenon. In certain regions the current for the first few cycles was that of normal exciting current which would then grow into the higher magnitudes of subharmonic response.

Subharmonics of the second, fourth, sixth, and ninth orders were also obtained, although no attempt was made to map the regions except to note that they were extremely small. It is of interest to note that the sixth- and ninth-order subharmonics were always accompanied by a predominant third-order subharmonic. Similarly, the fourth order was always accompanied by a predominant second order.

It was hoped that the results would be of such a type that definite conclusions could be reached regarding a means of avoiding ferroresonance and subharmonics by a proper initial charge on the capacitor. In the case of subharmonics no such conclusions could be drawn. It can be seen, however, that by putting zero charge on the capacitor, ferroresonance is not likely to be obtained whereas it may be obtained with a high value of initial charge. The placing of a resistor in parallel with the capacitor as suggested by Butler and Concordia<sup>8</sup> accomplishes this by allowing a path over which the residual charge may be dissipated.

## Appendix I

Frequency of applied voltage = 60 cycles per second

Series resistance  $R = 0.80$  ohm

Switching angle = 0 degree

Initial flux linkages = 0.0

Transformer number 1 = 2-kva distribution transformer, Hypersil core, 0.22-ohm conductor resistance

Transformer number 2 = power supply plate transformer, 2.20-ohm conductor resistance

Sign convention = switch closed with the voltage rise from  $a$  to  $b$ , see Figure 1(A), passing through zero from negative to positive

= charge on capacitor  $C$  arbitrarily defined as positive when  $d$  was positive with respect to  $e$

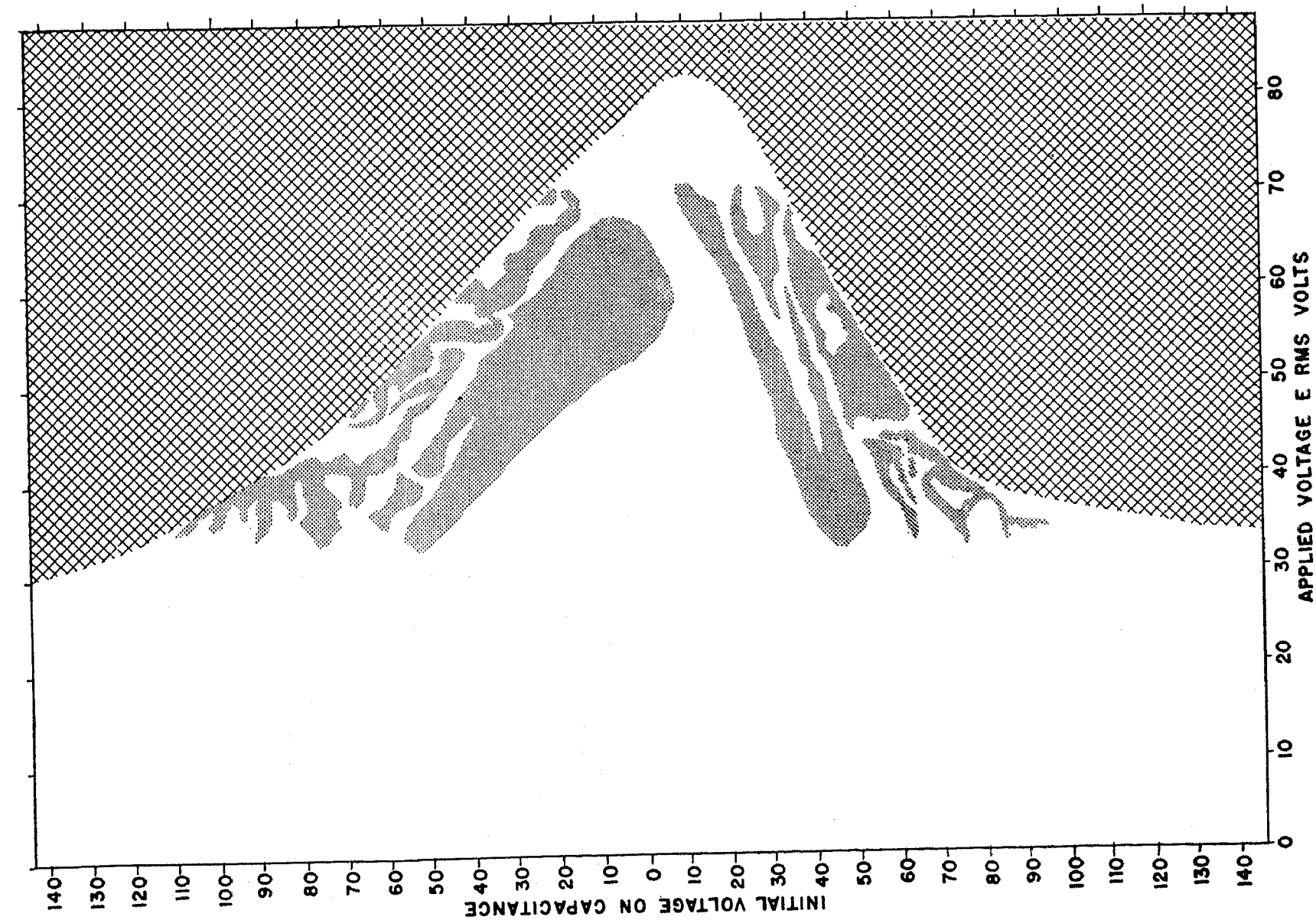


Figure 5. Areas of stable response. Transformer number 1—capacitance, 136.9 microfarads

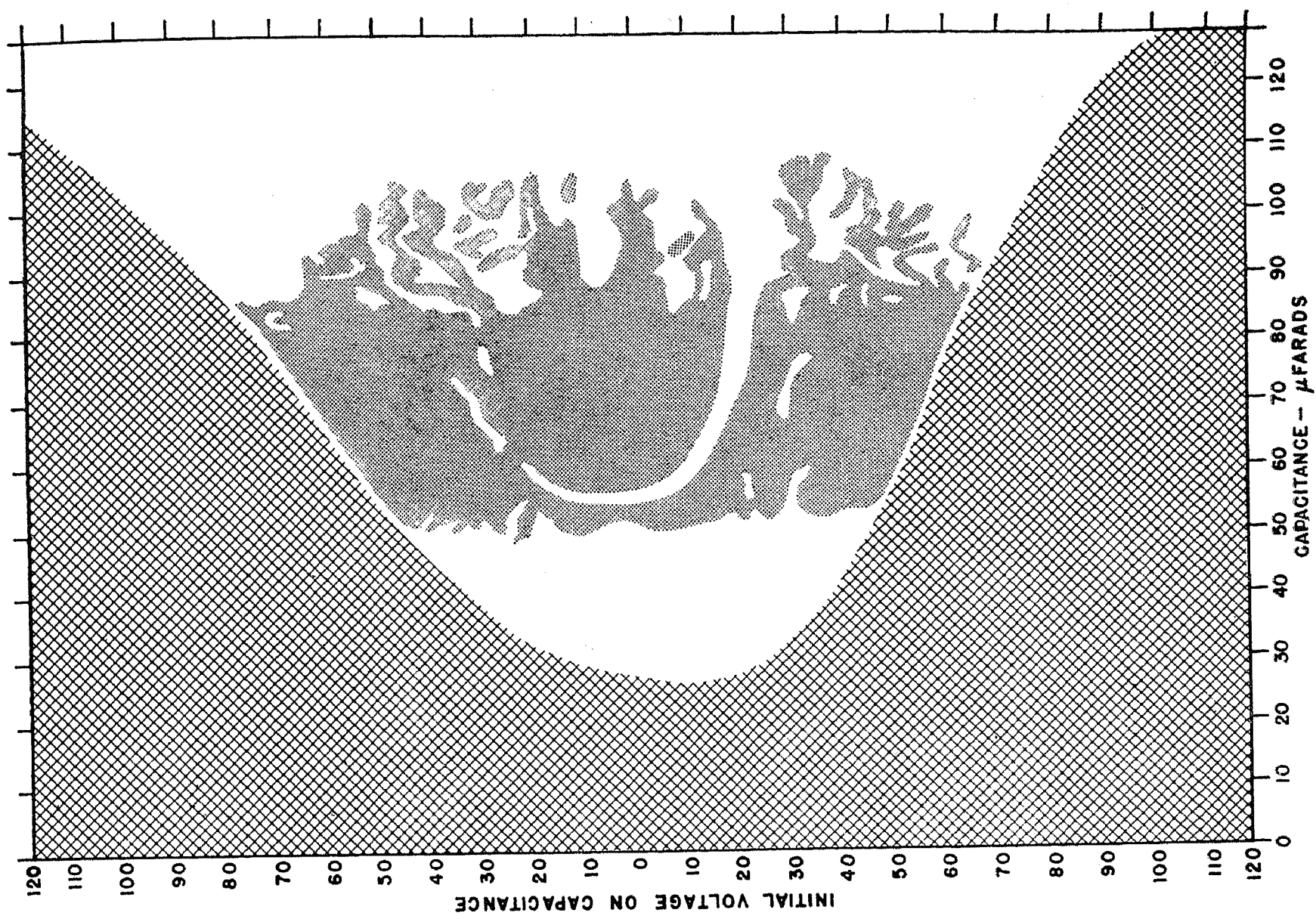


Figure 6. Areas of stable response. Transformer number 2—applied voltage, 55 volts rms

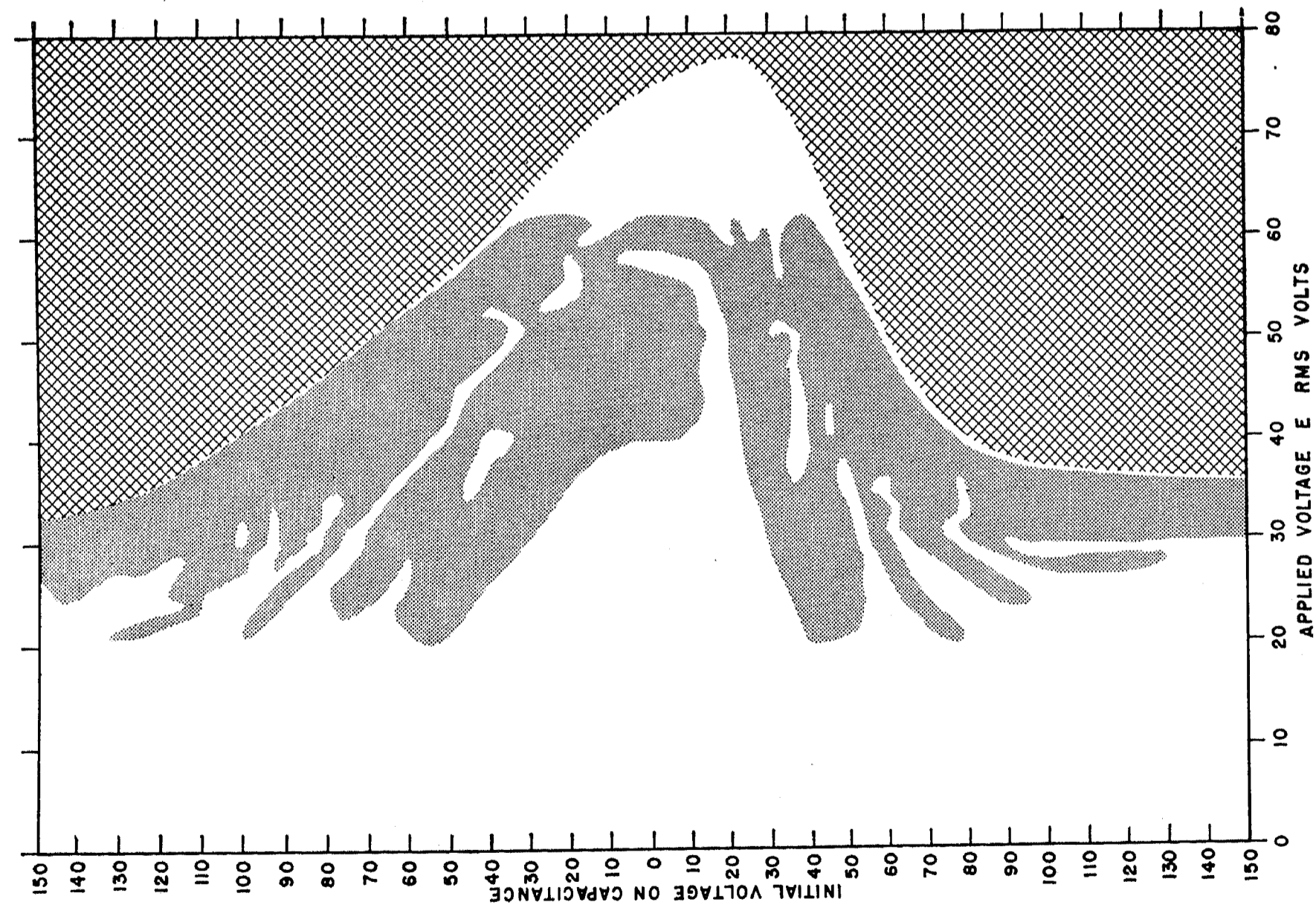


Figure 7. Areas of stable response. Transformer number 2—capacitor, 60.4 microfarads

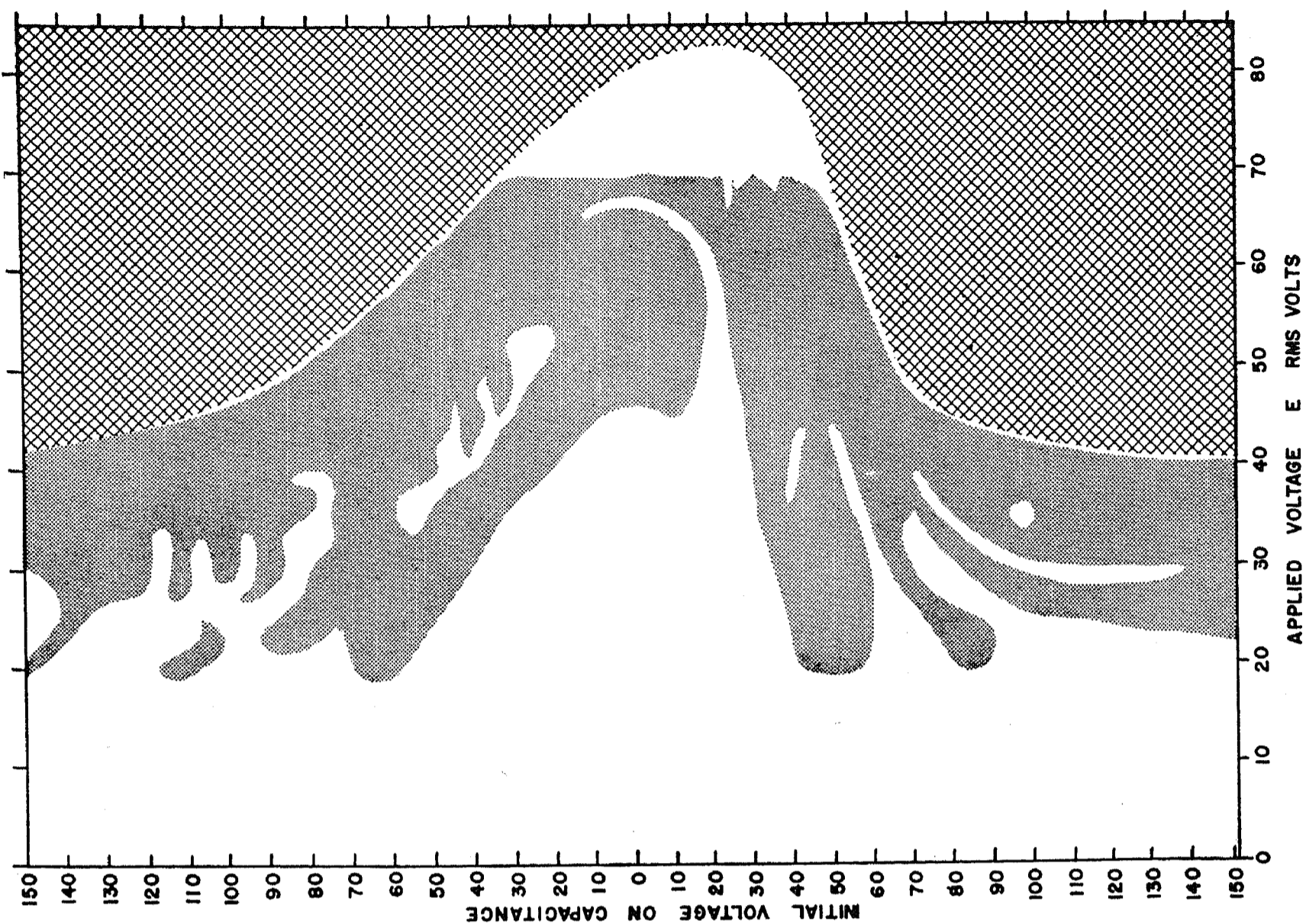


Figure 8. Areas of stable response. Transformer number 2—capacitor, 79.2 microfarads

In Figures 4 through 8, the area of stable third-order subharmonic response is represented by dots. The area of stable ferroresonant response is shown by crisscross lines and the area of normal exciting current is clear.

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### No Discussion

# Solution of Electrical Engineering Problems by Southwell's Relaxation Method

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**Synopsis:** The solution of simultaneous linear algebraic equations by the basic relaxation method is described in detail and the use of short-cut methods is explained and demonstrated. It then is shown that the same technique can be applied to the solution of ordinary and partial differential equations, and its accuracy is discussed. Several problems taken from various branches of electrical engineering are solved to illustrate the application of the method.

THE method of "relaxation" as devised by Southwell<sup>1-3</sup> for the solution of framework problems has proved to be of very wide application. Used first in the numerical solution of systems of simultaneous linear algebraic equations, it now has been extended, with outstanding success, to the solution of ordinary and partial differential equations. The technique is essentially the same for all equations, and many otherwise quite intractable problems have been solved by this method. Occasionally conditions are met in which the successive solutions converge only very slowly, or even not at all. This question of "ill-conditioning" in relation to the relaxation method was discussed by Fox.<sup>4</sup>

In electrical engineering the most likely applications of the method are to the solution of sets of simultaneous equations arising, for example, in the analysis of coupled circuits, and to the solution of

electrical and magnetic field problems, determined by the partial differential equations of Poisson and Laplace, or the wave equation. By other methods, these differential equations can be solved only in simple or in "idealized" cases, but relaxation can give a numerical solution for most boundary shapes and conditions.

Several accounts of the method and its application have been published<sup>1-7</sup> but few deal with the actual technique. The purpose of this paper is to present an outline of the basic principles of the method and to demonstrate the technique by examples taken from various branches of electrical engineering.

### Solution of Simultaneous Linear Algebraic Equations

#### BASIC PROCESS OF RELAXATION

Consider as an example of the method the pair of equations

$$79 - 3x + y = 0 \tag{1}$$

$$147 + x - 3y = 0 \tag{2}$$

Table I. Operations Table

	$\Delta R_1$	$\Delta R_2$
$\Delta x = 1$ .....	-3.....	1
$\Delta y = 1$ .....	1.....	-3

These equations may be written in the form

$$R_1 = 79 - 3x + y \tag{3}$$

$$R_2 = 147 + x - 3y \tag{4}$$

where, for the correct values of  $x$  and  $y$ ,  $R_1 = R_2 = 0$ . For any other values of  $x$  and  $y$ ,  $R_1$  and  $R_2$  are not zero.  $R_1$  and  $R_2$  are called the "residuals" and are a measure of the error of the solution for any given  $x$  and  $y$ .  $R_1$  and  $R_2$  can be calculated for any pair of  $x$  and  $y$ , but one wants to find that unique pair which makes  $R_1 = R_2 = 0$ .

First, an operations table is set up. From equations 3 and 4 it is seen that unit positive increment to  $x$  changes  $R_1$  by  $-3$  and  $R_2$  by  $+1$  and unit positive increment to  $y$  changes  $R_1$  by  $+1$  and  $R_2$  by  $-3$ . This is given in Table I.

One has to make an initial guess of  $x$  and  $y$  to start the computation (a good initial guess is helpful as it will shorten the subsequent numerical work, but is by no means essential). In this particular example, take for convenience  $x = 0$  and  $y = 0$  as initial values. From this one calculates the residuals  $R_1$  and  $R_2$  using equations 3 and 4 and enters these values in the relaxation table, Table II.

The biggest residual then is picked out, which is 147 in this case, and it reduces to

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Table II. Relaxation Table

	$R_1$	$R_2$
$x=0$ ..... $y=0$ .....	79.....	147
$\Delta y=49$ .....	128.....	0
$\Delta x=42$ .....	2.....	42
$\Delta y=14$ .....	16.....	0
$\Delta x=5$ .....	1.....	5
$\Delta y=1$ .....	2.....	2
$\Delta x=1$ .....	-1.....	3
$\Delta y=1$ .....	0.....	0
$x=48$ ..... $y=65$		
Check $R_1=79-144+65=0$ .		
$R_2=147+48-195=0$ .		

zero. Since  $\Delta y=1$  gives  $\Delta R_2=-3$ ,  $\Delta y=49$  gives  $\Delta R_2=-147$ , thereby making  $R_2$  zero, but at the same time it makes  $\Delta R_1=49$ , so increasing  $R_1$  to 128. This operation is entered in the second line of the Relaxation Table. Proceeding in the same way, take  $\Delta x=42$ , giving  $\Delta R_1=-126$ . This does not make  $R_1$  quite zero, but as near to zero as necessary. The corresponding increment to  $R_2=42$ , and this operation is written as line 3. The working is continued in this manner until both residuals are zero. (In line 7 a negative residual occurs, but this presents no difficulty.) The increments to  $x$  and  $y$  then are summed and the values so obtained substituted in the original equations, to check that  $R_1=R_2=0$ .

If a mistake had been made in Table II,  $R_1$  and  $R_2$  would not be found to be zero, but some other small numbers. These values then would be taken as new starting values for the residuals and "relaxed" to zero, the further increments required to do this being added to the previous ones to give the correct solution.

Relaxation is used to get a numerical solution as quickly as possible. The method involves a large number of very simple operations. Since the results can be checked easily during the progress of the work, it is better to work quickly and risk arithmetical errors, making checks of the residuals only occasionally, than to stop and check each step by which the residuals have been calculated. To make the method even more speedy, several shortening devices have been developed.

#### SHORTENING DEVICES

*Overrelaxation.* From the example it can be seen that whenever one residual is decreased the other is increased though by a smaller amount. If, therefore, instead of making just sufficient increment to make a residual zero, one makes a larger increment, that is, one overrelaxes to make the residual negative, then when it receives an increment as a result of another operation it still will be near zero.

This can be seen from the alternative method, Table III, of working the previous example.

Table III. Alternative Method

	$R_1$	$R_2$
$x=0$ ..... $y=0$ .....	79.....	147
$\Delta y=60$ .....	139.....	-33
$\Delta x=45$ .....	4.....	12
$\Delta y=5$ .....	9.....	-3
$\Delta x=3$ .....	0.....	0
$x=48$ ..... $y=65$		

Table III, of course, gives the same answer but it took fewer lines of working.

*Block Relaxation.* By definition this is the application of unit increment to more than one of the variables at a time. That is a superposition of the effects  $\Delta x=1$  and  $\Delta y=1$ , giving a new line in the operations table shown in Table IV.

Table IV

	$\Delta R_1$	$\Delta R_2$
$\Delta x=1$ .....	-3.....	1
$\Delta y=1$ .....	1.....	-3
$\Delta x=\Delta y=1$ .....	-2.....	-2

When the total residual is zero, individual residuals must be evenly distributed among positive and negative residuals, and such residuals are much easier to "liquidate" than a set which all have the same sign. Block relaxation is very useful in reducing the total residual (that is, the sum of all residuals) to zero. To demonstrate this the same example is again solved.

Table V

	$R_1$	$R_2$
$x=0$ ..... $y=0$ .....	79.....	147
$\Delta x=56$ ..... $\Delta y=56$ .....	-33.....	35
$\Delta y=9$ .....	-24.....	8
$\Delta x=-8$ .....	0.....	0

*Group Relaxation.* By definition this is the application of unequal increments to more than one of the variables at a time.

Any new line can be formed in the operations table to meet a particular requirement; an example is given in Table VI.

This particular operation would be used if one residual were very large compared with the other.

*Equalization of Residuals.* In any set of linear equations, if the initial residuals are  $R_0(x)$ ,  $R_0(y)$ ,  $R_0(z)$  ... it is possible by ap-

plying appropriate increments  $\Delta_n x$ ,  $\Delta_n y$ ,  $\Delta_n z$  ... to reduce these values to the values  $R_n(x)$ ,  $R_n(y)$ ,  $R_n(z)$  ... such that

$$\frac{R_n(x)}{R_0(x)} = \frac{R_n(y)}{R_0(y)} = \dots = K (0 < K < 1)$$

It is then found that further increments  $(K/1-K)\Delta_n x$ ,  $(K/1-K)\Delta_n y$ ,  $(K/1-K)\Delta_n z$  ... will reduce all the residuals to zero.

#### APPLICATION OF THE METHOD TO COUPLED CIRCUITS

For the general case of a  $p$ -mesh linear network, the steady state is described by the  $p$  equations

$$E_1 = Z_{11}I_1 + Z_{12}I_2 + \dots + Z_{1p}I_p$$

$$E_2 = Z_{21}I_1 + Z_{22}I_2 + \dots + Z_{2p}I_p$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$E_p = Z_{p1}I_1 + Z_{p2}I_2 + \dots + Z_{pp}I_p$$

where

$E$ 's are impressed voltages (constant)

$Z$ 's are impedances (constant)

$I$ 's are mesh currents (unknown)

The currents then can be found by solving this system of simultaneous linear algebraic equations. While the solution can be obtained by elimination, it is found that in most cases relaxation is simpler and quicker to carry out.

Relaxation has the added advantages that

1. It very quickly gives an approximate solution which may be all that is required.
2. Starting from this approximate solution, a solution of any desired accuracy can be obtained.
3. The progress of the solution can be checked easily by recalculating the residuals from the original equations.

Its disadvantage is that all the currents must be determined simultaneously whether they are all of interest or not.

#### APPLICATION OF THE METHOD TO CHARACTERISTIC VALUE PROBLEMS

Systems of simultaneous equations dealing with characteristic value problems, for example, vibrating engineering structures, can be solved by a combination of the relaxation process as described, and of Rayleigh's principle. This method has been discussed fully in a paper by Fox<sup>4</sup>

Table VI

	$\Delta R_1$	$\Delta R_2$
$\Delta x=1$ .....	-3.....	1
$\Delta y=1$ .....	1.....	-3
$\Delta x=2$ ..... $\Delta y=1$ .....	-5.....	-1

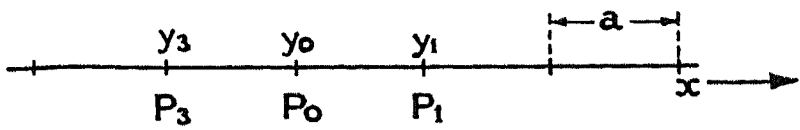


Figure 1. Subdivision of range into finite intervals

in which examples are given showing how certain devices can be used to shorten the process, and also how modes other than the lowest can be found.

## Solution of Ordinary Differential Equations

### NATURE OF THE PROBLEM

Ordinary differential equations of the second order can be written in the form

$$\frac{d^2y}{dx^2} = z\left(x, y, \frac{dy}{dx}\right)$$

where  $z$  is a known function of  $x$ ,  $y$ , and  $dy/dx$ . For a unique solution of the equation two boundary conditions also must be specified. Sometimes the forms of the equation and boundary conditions are such that an analytical solution is impossible and then numerical methods must be used. Usually, either  $y$  and  $dy/dx$  are known at one end of the range of solution or  $y$  is known at each end. In the first of these cases, an initial-value problem, the problem is usually solved by step-by-step methods, while in the second case, a boundary-value problem, relaxation is a much better method.

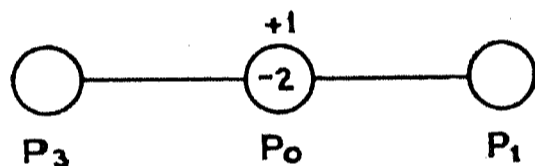


Figure 2. Incomplete relaxation pattern (first step)

Although differential equations require for their solution a continuous function in the given range, relaxation, like all other numerical methods, gives only numerical values at a series of points. These points are taken for convenience as equidistant points along the  $x$ -axis, and values at intermediate points then can be found by graphical or algebraic interpolation.

Relaxation cannot solve differential equations as such, but by replacing them

by the corresponding finite-difference equations they can be converted into systems of simultaneous linear algebraic equations which are amenable to the standard relaxation techniques, provided that the solution does not contain singular points within the range of the solution.

Differential equations leading to characteristic values and characteristic solutions can be dealt with as mentioned previously in the section entitled "Application of the method to characteristic value problems" after the differential equation has been replaced by a set of simultaneous linear equations.

### FINITE DIFFERENCE APPROXIMATION

Consider, for example, the equation

$$\frac{d^2y}{dx^2} + K \frac{dy}{dx} - z(x) = 0$$

with the boundary conditions that  $y$  is known at each end of the range of  $x$  under consideration. This equation, with  $K$  being a constant, of course can be solved directly and the appropriate boundary conditions inserted, but it was chosen here as an example to demonstrate the method in a simple way. One divides this range of  $x$  into equal mesh lengths  $a$ . Taking any nodal point as the origin  $P_0(x_0)$  where  $y$  has the value  $y_0$ , and adjacent points  $P_1(x_0+a)$  and  $P_3(x_0-a)$  where  $y=y_1$  and  $y=y_3$  as shown in Figure 1, one can show from Taylor's expansion that the finite difference approximations are

$$\left(\frac{dy}{dx}\right)_0 = \frac{y_1 - y_3}{2a}$$

and

$$\left(\frac{d^2y}{dx^2}\right)_0 = \frac{y_1 + y_3 - 2y_0}{a^2}$$

The given equation therefore can be approximated by

$$y_1(2+ak) + y_3(2-ak) - 4y_0 - 2a^2z_0 = 0$$

This is a linear algebraic equation, and since an equation of this form connects

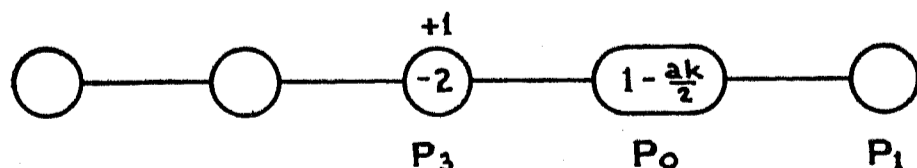


Figure 3. Incomplete relaxation pattern (second step)

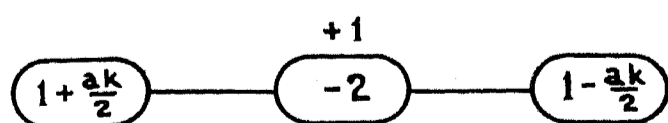


Figure 4. Complete relaxation pattern

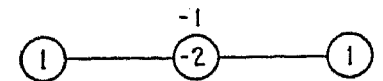


Figure 5. Relaxation pattern for differential equation  $y'' - z(x) = 0$

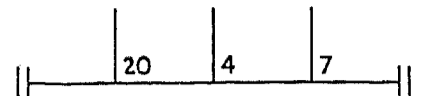


Figure 6. Assumed initial values of residuals

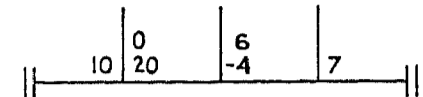


Figure 7. Increments and resulting residuals after first relaxation step

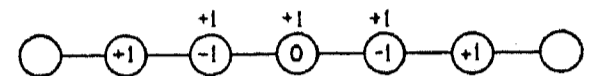


Figure 8. Relation pattern for block relaxation

every set of three consecutive points, the given differential equation has been replaced by a set of simultaneous linear algebraic equations, the number of equations in the set being given by the number of points at which the solution is determined.

The residual  $R_0$  at the point  $P_0$  is given by

$$R_0 = y_1(1+ak/2) + y_3(1-ak/2) - 2y_0 - a^2z_0 \quad (5)$$

### BASIC PROCESS OF RELAXATION

Since all the equations 5 (for different points  $P_0, P_1, \dots, P_n$ ) are of the same form, every line of the operations table is of the same form. It is best visualized not as a table but as a "relaxation pattern."

This relaxation pattern is a graphical representation of the effect on residuals at various points of unit increment at any given point. Patterns can be formulated for any residual formula, and they are the essential feature of the relaxation method. Once the pattern has been made, it can be applied to the numerical work in which all the residuals are reduced to negligible values. It is left to the discretion of the computer to decide at which points these patterns will be used most profitably. This makes relaxation a more interesting and more flexible computational technique than the purely mechanical iteration methods.

The character of the relaxation pattern depends on the form of the given differential equation. It may be the same for all points in the region, or it may depend on the position of the origin. The method of deriving the pattern, however, is the same

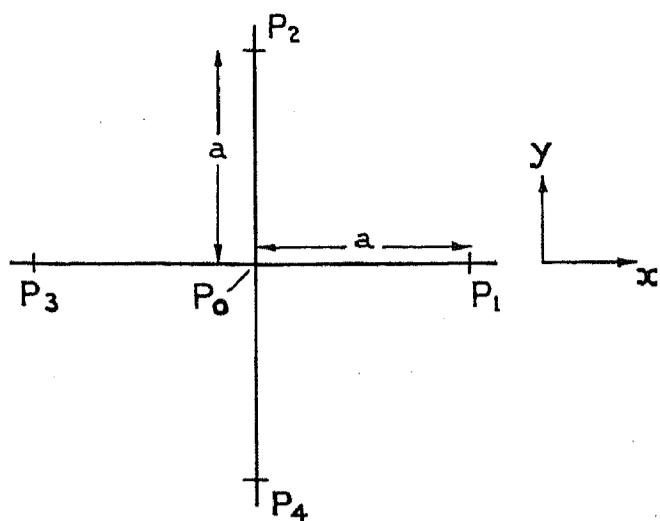


Figure 9. Mesh points surrounding point  $P_0$ .

in all cases. As an example, consider the formulation of the pattern corresponding to equation 5.

It is seen from equation 5 that  $\Delta R_0 = -2$  for  $\Delta y_0 = +1$ . This is represented graphically by the (incomplete) pattern, Figure 2, the increment  $\Delta y$  being written above the circle, the resulting change of residual inside the circle. However, the

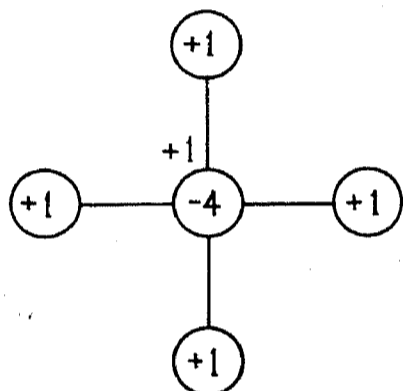


Figure 10 (left). Relaxation pattern for Laplace's equation

Figure 13 (right). Potential values, increments, and residuals after further relaxation

increment  $\Delta y_0$  does not only change the residual at the point  $P_0$  but also the residuals  $R_1$  and  $R_3$  at the neighboring points  $P_1$  and  $P_3$ . The changes of the residuals at these points due to the increment  $\Delta y_0 = 1$  are obtained from equation 5 by changing the origin first to  $P_1$  and then to  $P_3$ . Taking the new origin at  $P_1$ , the value  $y_0$  takes the place of  $y_3$  in equation 5, and hence the increment  $\Delta y_0$  produces in the residual  $R_1$  the change  $\Delta R_1 = (1 - ak/2)\Delta y_0$ . This is shown graphically in Figure 3 for a unit increment  $\Delta y_0 = 1$ . Similarly one finds that the change in the residual  $R_3$  is  $\Delta R_3 = (1 + ak/2)\Delta y_0$ . The resulting

13	20	40	80	76			
8	16	-1	30	-9	50	-20	60
5	5	8	5	41	10	83	50
4	2	25	16	-36	18	29	45
7	4		3	30			40

Figure 11 (left). Initial potential values and residuals for solution of Laplace's equation

Figure 14 (right). Subdivision of relaxation nets

13	20	40	80	76			
8	16	-1	30	-9	50	20	60
5	5	8	5	41	10	83	50
4	2	25	16	-36	18	29	45
7	4	3	30				40

Figure 12. Potential values, increments, and residuals after first relaxation step

complete relaxation pattern, representing the change of all residuals due to a unit increment of  $y$  at  $P_0$ , is shown in Figure 4.

Consider now the simple example where  $K=0$ . Here the relaxation pattern reduces to that shown in Figure 5. Suppose the residuals are initially as shown in Figure 6. As in the case of simultaneous equations, one starts by picking out the largest residual 20. From the relaxation

13	20	40	80	76			
			6	0			
8	16	-1	30	-9	50	-20	60
			1				
5	5	23	15	61	20		50
		8	5	41	10	83	
4	2	25	16	-21	18	49	45
				-36		29	
7	4		3	30			40

range are given by fixed boundary conditions and so there can at no time be any residual there.) The process is continued until all the residuals may be deemed negligible, for the accuracy desired. It is also necessary to impose the condition that the total residual shall be small.

#### SHORTENING DEVICES

As before, overrelaxation is quite straightforward and very useful.

Patterns for block relaxation are formed by superposing individual patterns and are seen to be of the form shown in Figure 8.

Patterns for group relaxation can also be formed to suit any particular case; see the section on "Numerical Example" given later in the paper.

#### ACCURACY OF THE METHOD

The difference equations can be solved to any degree of accuracy but the ultimate accuracy of the solution depends on the closeness of the approximation of these difference equations to the given differential equation. Since terms of order  $a^3$  were neglected in this approximation, the solution can be accurate only when worked for values of  $a$  which are small enough for these terms to be truly negligible. The usual practice is to work solutions for  $a = (x_2 - x_1)/2n$  ( $n=1,2,3 \dots$ ) where  $a$  is the mesh length and  $x_2 - x_1$  the complete range of solution, with  $n$  increasing until two subsequent solutions differ by a negligible amount. Even though initial solutions worked for a large value of  $a$  may be quite far from the final, correct solution, they are useful and time-saving in giving a good initial guess when the interval is halved, and the convergence of the solution can be seen.

As an example, the equation

$$\frac{d^2y}{dx^2} + x(1-x)10^4 = 0$$

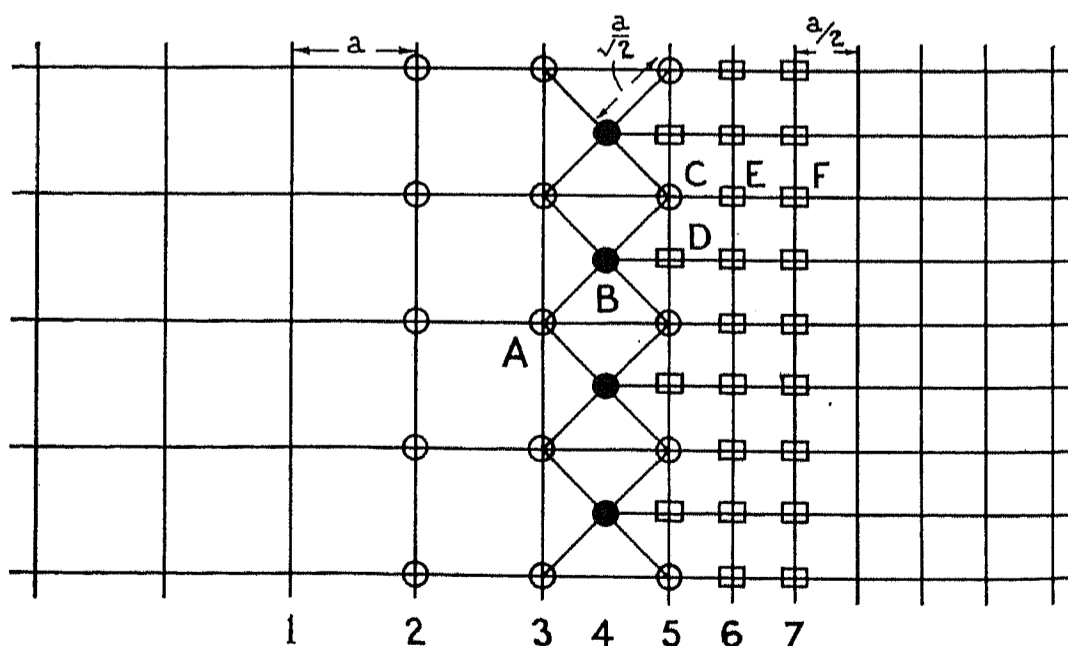


Table VII

n	1	2	3	4	Extra- polated Value	Analyt- ical Solution
y	313	273	264	261	260	260.4

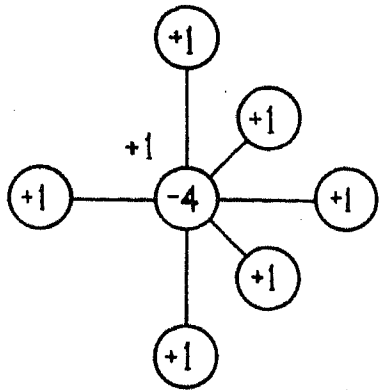


Figure 15. Relaxation pattern at point A in Figure 14

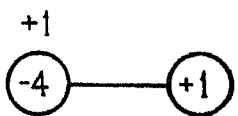


Figure 16. Relaxation pattern at point B in Figure 14

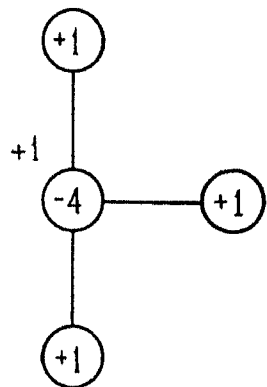


Figure 17. Relaxation pattern at point E in Figure 14

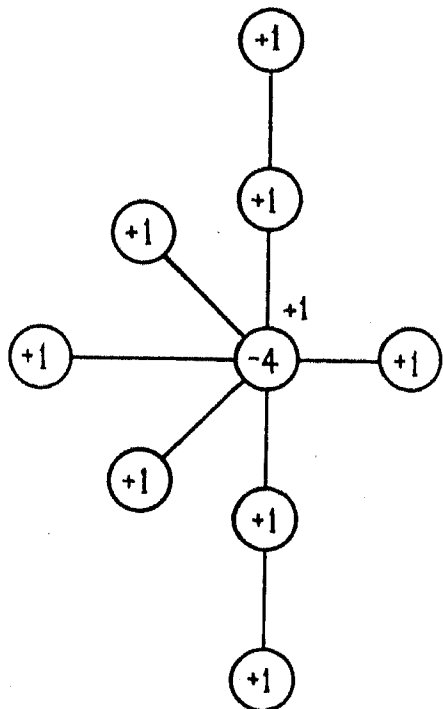


Figure 18. Relaxation pattern at point C in Figure 14

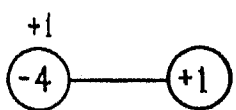


Figure 19. Relaxation pattern at point D in Figure 14

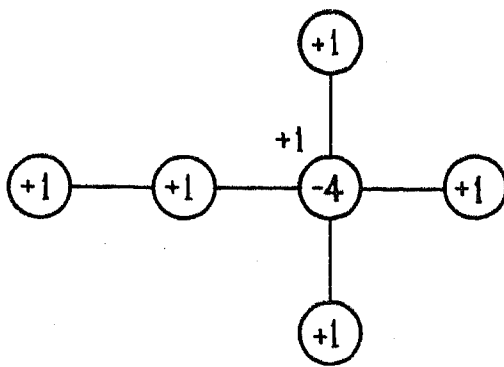
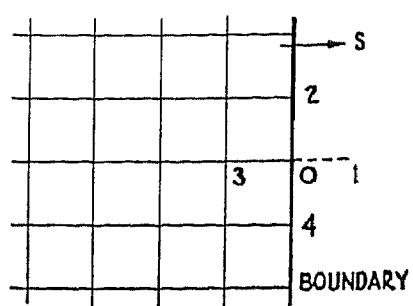


Figure 20 (left). Relaxation pattern at point F in Figure 14

Figure 22 (right). Fictitious boundary point to represent prescribed boundary gradient



with  $y=0$  at  $x=0$  and  $y=0$  at  $x=1$ , which has the analytical solution  $y=260.4$  at  $x=1/2$ , has been solved for various  $n$  and the results are shown in Table VII.

Often, the solution approaches upon repeated reduction of the interval size  $a$  the correct value in an oscillatory manner and extrapolation must be carried out with care as was pointed out by Salvadori.<sup>8</sup>

### Solution of Poisson's and Laplace's Equations in Two Dimensions

#### NATURE OF THE PROBLEM

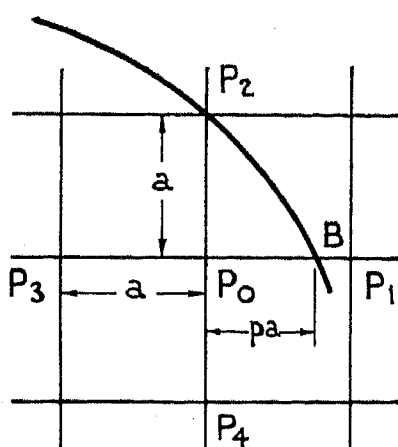
Partial differential equations may also be of initial-value or boundary-value types. As in the case of ordinary differential equations it is the boundary-value types which are solved most conveniently numerically by relaxation. Such problems arise frequently in electrical engineering. For instance, in electric machines, the shape of the boundary and the field values on the boundary are known, and the field configuration inside the region is required. Here, electric and magnetic potentials satisfy Poisson's or Laplace's equations.

The first step in solving these equations is to replace them by their finite difference approximations, that is, a set of simultaneous linear algebraic equations holding at various points at which the solution is to be obtained. The most convenient points to take are the nodes of a symmetric and systematic network; in practice, square meshes generally are used, though triangular and hexagonal meshes are geometrically possible.

#### FINITE DIFFERENCE APPROXIMATION

Poisson's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = z(xy)$$



may be written in finite difference form as

$$\sum_{n=1}^4 V_n - 4V_0 = a^2 z_0$$

$a$  being the mesh size (see Figure 9) whence the residual  $R_0$  at the point  $P_0$  is given by

$$R_0 = \sum_{n=1}^4 V_n - 4V_0 - a^2 z_0 \quad (6)$$

For Laplace's equation  $z=0$ , and the equation reduces to

$$R_0 = \sum_{n=1}^4 V_n - 4V_0 \quad (7)$$

#### BASIC PROCESS OF RELAXATION

By arguments similar to those in the preceding section of the same name, it is found that the appropriate relaxation pattern in this case is that shown in Figure 10.

In solving Laplace's equation, for given boundary values, in the region shown in Figure 11,  $V$  values first are guessed and entered to the left of the nodes, the calculated residuals being entered to the right.

The largest residual then is relaxed as shown in Figure 12, the process continued as in Figure 13 and further continued until

1. All the residuals are small.
2. The total residual is small.
3. The total residual in any area is small.

Finally, all increments  $\Delta V$  are summed for each node.

#### SHORTENING DEVICES

Any device which will increase the speed of the method is of great importance

Figure 21 (left). Unequal network star for curved boundary

Figure 23 (below). Section through instrument magnet

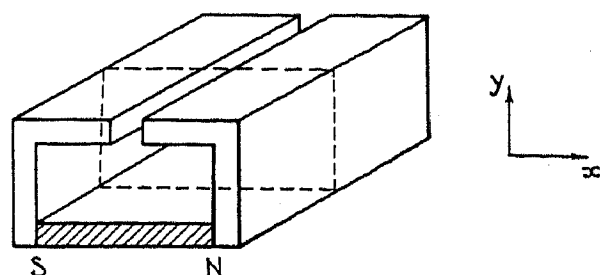
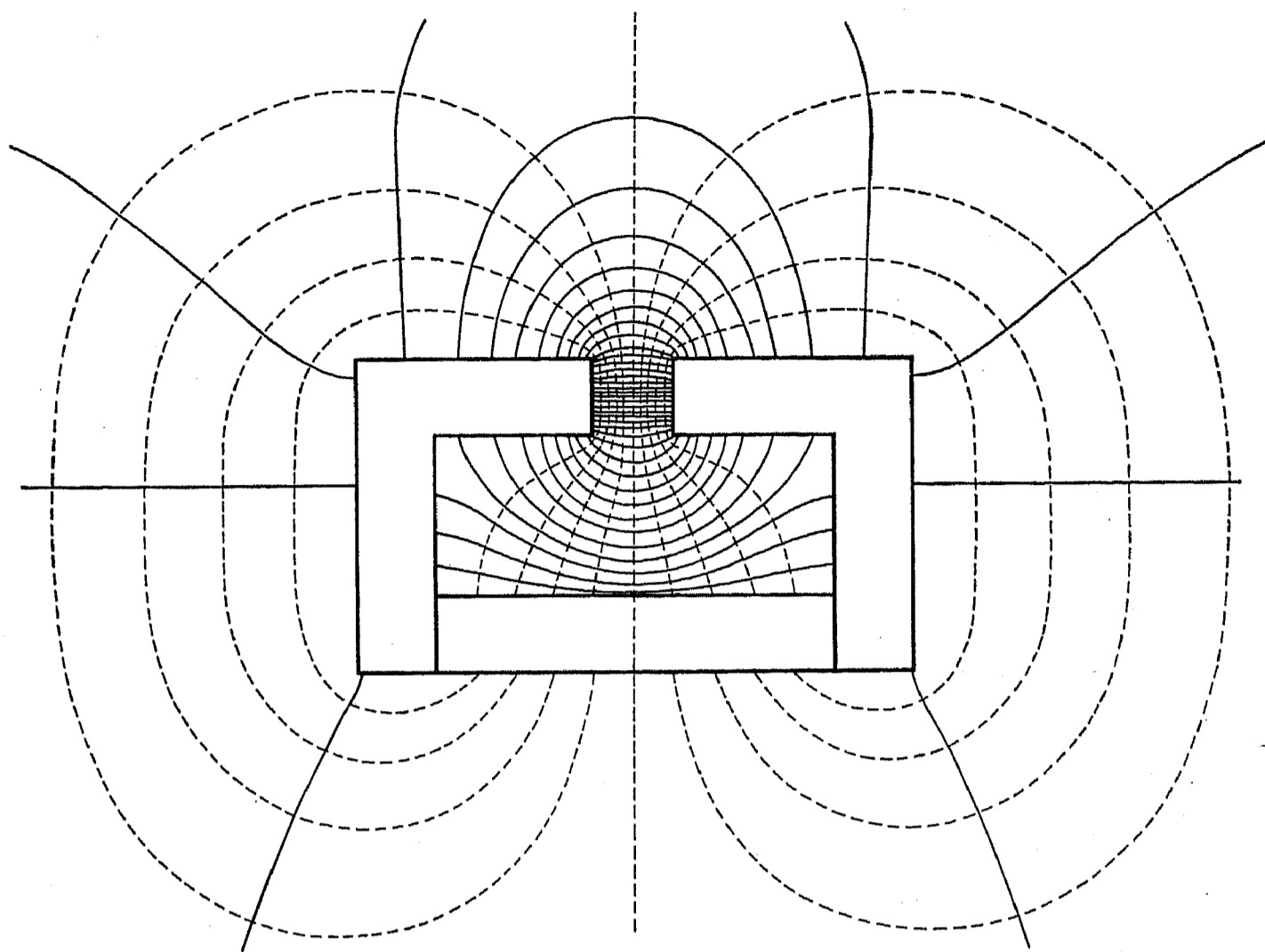




Figure 26. Magnetic flux lines and equipotentials of magnetomotive force for magnet, Figure 23, as finally computed



to use a fine mesh only where necessary, but one then has the problem of joining the two types of net.

In Figure 14 a net of mesh of size  $a$  is shown, which is to be halved in the right-hand region. The procedure is as follows:

1. Draw in the extra mesh lines required.
2. Guess values at the new nodes so formed.
3. Calculate the residuals at these points.

To be able to calculate the residual at a point, the value of the function has to be known at that point and also at the symmetrically surrounding points. Some of the nodes just formed do not have symmetrically surrounding points and so present a special problem. Referring to Figure 14:

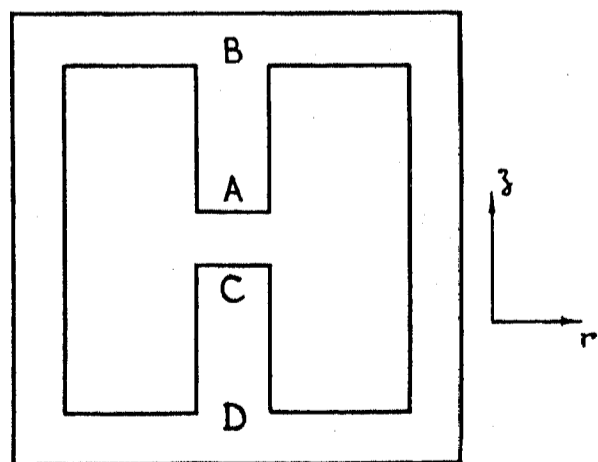


Figure 27. Section through cylindrical air-gap choke

Columns 1, 2, 3, and to the left—these are all in the coarse mesh.

Columns 6, 7, and to the right—these are all in the fine mesh.

Column 5—alternate points can be regarded as being in the coarse mesh, but the others

have no point to their left. Produce the fine net one step further at these places to get the points shown in column 4.

Column 4—these points are now surrounded by points distance  $a/\sqrt{2}$  away. The residuals are now calculated from equations 6 or 7 but with  $a$  replaced by  $a/\sqrt{2}$ . (This

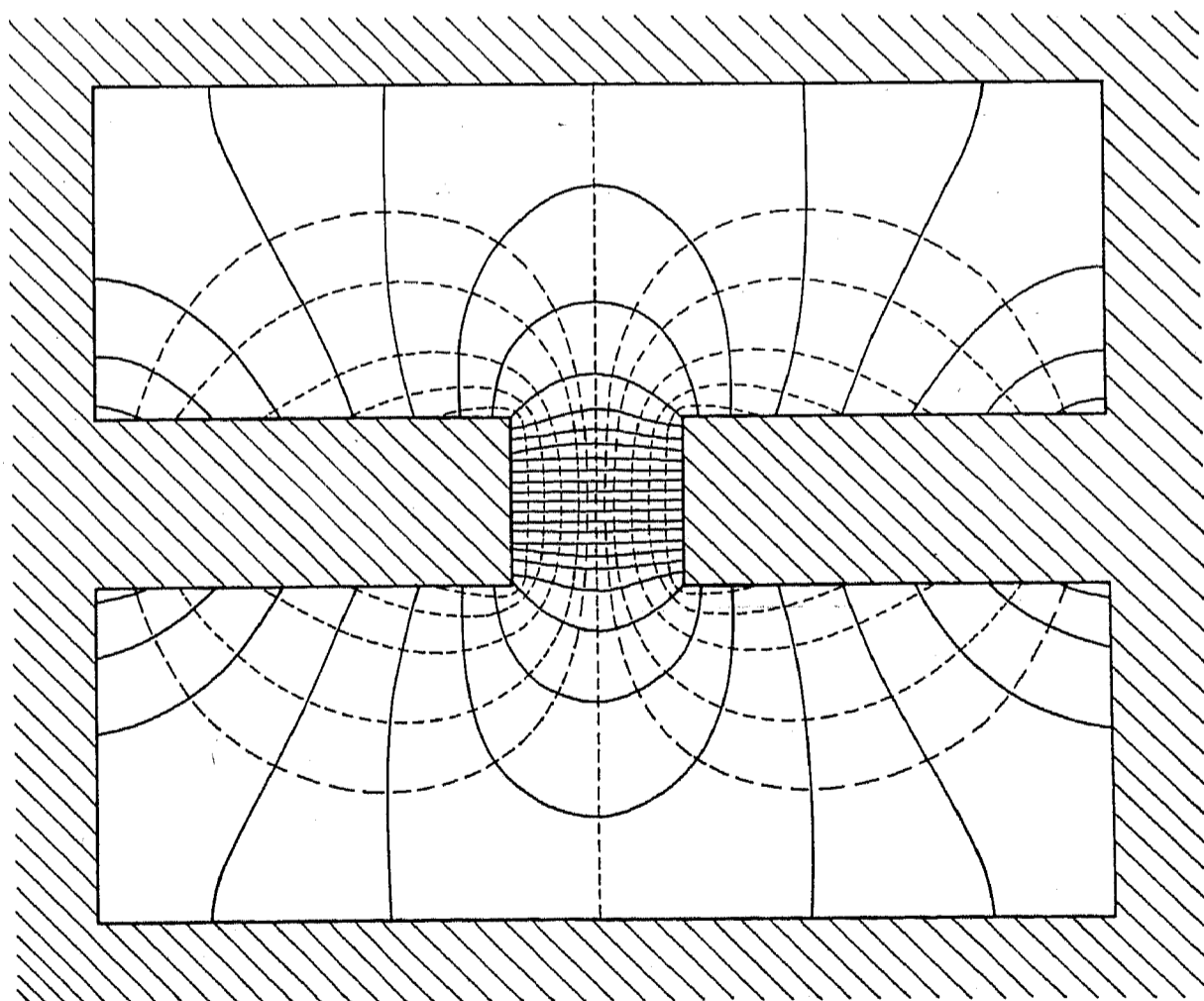


Figure 28. Magnetic flux lines and equipotentials of magnetomotive force for air-gap choke, Figure 27, as computed by relaxation

is permissible since  $\nabla^2$  remains invariant under rotation of axes.)

The residuals at all points now can be calculated. In the region of the join, however, modified relaxation patterns also are required. Points  $A, B, C, D, E$ , and  $F$  in Figure 14 are typical points in the region.

Consider the point  $A$ . Changing the value there by  $+1$  means changing its residual by  $-4$  units. Now consider which other points are affected by this change. The value at  $A$  is used in the calculation of the residuals at all the points surrounding it on the large mesh, and also at the points above and below it on the diagonal mesh. Thus, the complete pattern at  $A$  is that shown in Figure 15.

Similarly, the patterns at points  $B, C, D, E$ , and  $F$  are found to be those of Figures 16 to 20.

Now these patterns can be used to relax in both regions and across the join.

It is sometimes possible to dispense with the diagonal mesh, but then relaxation can only be carried out in each region separately and residuals are liable to build up on the boundary.

*Inclusion of Higher Order Terms.* Fox<sup>9</sup> has shown that by taking account of higher order terms in the finite difference approximation greater accuracy can be obtained with a relatively large value of the mesh length  $a$ .

Since the work involved increases so rapidly with  $a$ , this method of computation is often very useful. The calculation of the required differences, however, may be a little tedious.

#### CURVED BOUNDARIES

If the region considered has curved boundaries, it is impossible to draw a square mesh to fit it exactly. There is then the problem of "unequal stars" as in

Figure 21. In this case there is no means of knowing  $V_1$  (at point  $P_1$ ), but  $V_B$ , the potential at the boundary point  $B$ , is known as part of the boundary condition. Also, the length  $pa$  can be calculated.

From a Taylor expansion first with  $x=a$ , and then  $x=-a$ , equation 8 for the residual  $R_0$  at  $P_0$  is obtained.

$$R_0 = V_2 + V_4 + \frac{2V_3}{p+1} + \frac{2V_B}{p(p+1)} - V_0 \left( 2 + \frac{2}{p} \right) - a^2 z_0 = 0 \quad (8)$$

In the case where two lines are cut by the boundary at distances  $pa$  and  $qa$  from  $P_0$ , the corresponding residual equation 9 is

$$R_0 = \frac{2V_3}{p+1} + \frac{2V_4}{q+1} + \frac{2V_B}{p(p+1)} + \frac{2V_C}{q(q+1)} - V_0 \left( \frac{2}{p} + \frac{2}{q} \right) - a^2 z_0 = 0 \quad (9)$$

In obtaining these equations it was necessary to neglect terms of order  $a^3$ , making them inherently less accurate than the standard equations 6 or 7.

#### BOUNDARY CONDITIONS

A slightly modified procedure is necessary if not the function, but its normal derivative is known at the boundary.

Consider first a rectilinear boundary as shown in Figure 22, with  $S$  the direction normal to the boundary,  $\partial V/\partial S$  being

known. For all points on the boundary,  $V_1$  would have a fictitious value, and hence the standard equation cannot be applied directly. However, one may say

$$\frac{V_1 - V_3}{2a} = \left( \frac{\partial V}{\partial S} \right)_0$$

hence  $V_1$  may be found in terms of known

values and substituted in the standard equation.

When the boundary is curved, two of the required values may be fictitious, and the normal to the boundary is not in general parallel to the mesh lines. This makes it a little more difficult to express the fictitious values in terms of known ones, but it is still quite straightforward. This method of attack is due to Fox.<sup>10</sup> An alternative approach due to Southwell

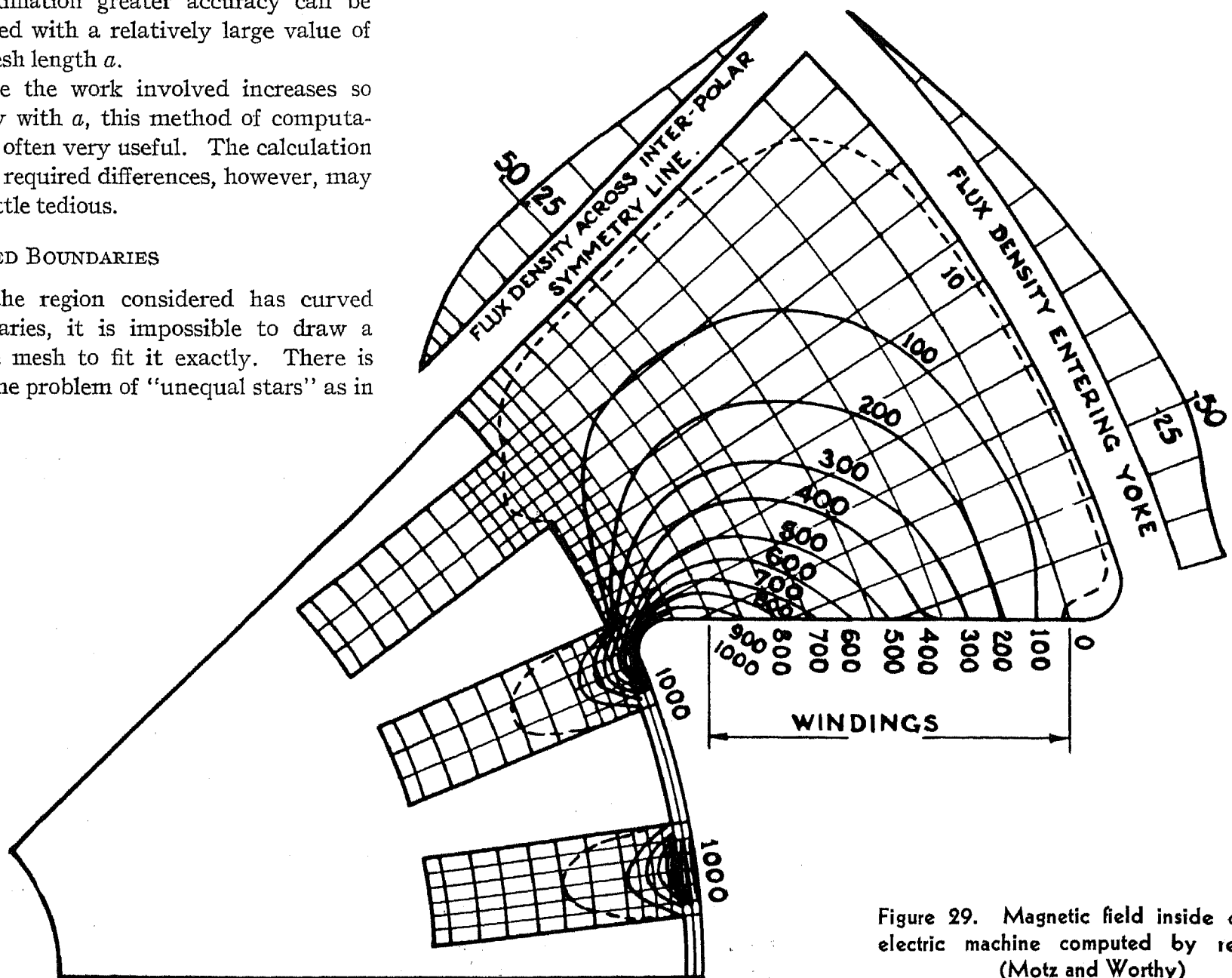


Figure 29. Magnetic field inside dynamo-electric machine computed by relaxation (Motz and Worthy)

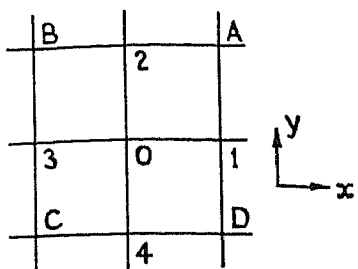


Figure 30 (left). Mesh points surrounding point 0, used in finite difference approximation of "mixed" derivatives

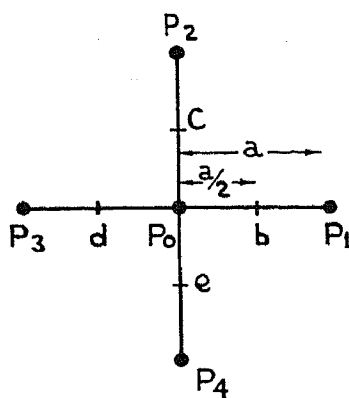


Figure 32 (right). Mesh points used in finite difference approximation of "quasi-plane potential equation"

and Vaisey<sup>11</sup> is based on the membrane analogy. This leads to different equations, but Fox has shown that the agreement between solutions worked by either method is quite good.

#### NUMERICAL EXAMPLE

Consider as an example the bar magnet with pole pieces as shown in Figure 23. This is similar to the type used in measuring instruments, and so is of frequent occurrence. The problem is really a 3-dimensional one, but if the width is reasonably large compared with the other dimensions, a section through a mid-plane can be regarded as experiencing no transverse end-effects, and can be treated as a 2-dimensional case. This means the governing equation has the form

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0$$

where  $H = -\text{grad } \Psi$ , being the magnetic potential and  $H$  the field strength. The shape of the boundary is specified accurately, but the values of  $H = -\text{grad } \Psi$  on the boundary are not known exactly. As a first approximation it is assumed that there is a linear drop of magnetic potential  $\Psi$  along the magnet and that the iron has infinite permeability.

Working on a net of mesh length one-quarter the gap width, the residuals were found to be predominantly negative giving a large value for the sum of the residuals. The following device was found particularly useful in reducing this total residual. The total residual in each column was calculated and, the number of points in each column being known, it was possible to make patterns for relaxing the whole column at once. In this way the distribution shown in Figure 24 was obtained. Smaller meshes were used then, as shown in Figure 25, and since the residuals on the boundaries of each mesh were found to change very little, diagonal nets for joining the meshes were unneces-

sary. Equipotentials and flux lines are shown in Figure 26.

#### Solution of 3-Dimensional Partial Differential Equations

Although the finite difference equations can be formulated easily for 3-dimensional problems, the number of points involved and the difficulties in recording make the method impracticable in general.

Several special cases of 3-dimensional partial differential equations, however, can be treated in this way. Tranter,<sup>12</sup> for instance, has shown how to reduce the number of independent variables in a given partial differential equation by one by using Fourier transforms, thus reducing 3-dimensional equations to 2-dimensional form. Also, the important class of equations having rotational symmetry can be written in essentially 2-dimensional form.

##### AXIALLY SYMMETRIC FIELDS

Laplace's equation in this case is

$$\frac{\partial^2 V}{\partial r^2} + \frac{\partial^2 V}{\partial z^2} + \frac{1}{r} \frac{\partial V}{\partial r} = 0$$

whence we have the equation for the residual

$$R_0 = V_1 + V_2 \left(1 - \frac{a}{2r}\right) + V_3 + V_4 \left(1 + \frac{a}{2r}\right) - 4V_0 \quad (10)$$

where  $r \neq 0$ . As

$$\lim_{r \rightarrow 0} \left( \frac{1}{r} \frac{\partial V}{\partial r} \right) = \frac{\partial^2 V}{\partial r^2}$$

One finds for the residual at a point  $P_0$  on the axis

$$R_0 = V_1 + V_3 + 4V_4 - 6V_0 \quad (11)$$

where  $r = 0$ .

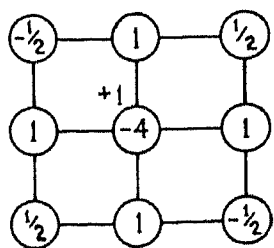
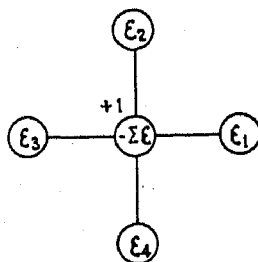


Figure 31 (left). Relaxation pattern for equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + 2 \frac{\partial^2 V}{\partial x \partial y} = 0$$

Figure 33 (right). Relaxation pattern for "quasi-plane potential equation"



#### NUMERICAL EXAMPLES

*Cylindrical Air-Gap Choke.* Consider the air-gap choke shown in Figure 27. Assuming as boundary conditions a linear drop of magnetic potential drop along  $AB$  and  $CD$ , and constant potential on the rest of the boundary, relaxation was performed on successively finer nets, as described in the previous section on "Graded Networks." The equipotentials then were plotted and these with the flux lines are shown in Figure 28.

*Dynamo-Electric Machines.* As shown in a paper by Motz and Worthy<sup>13\*</sup> the magnetic field inside a dynamo-electric machine can be calculated by this method. Their result for a 4-pole machine with a slotted armature and distributed windings is shown in Figure 29.

*Electron Lenses.* Most electron lenses have axial symmetry and hence the governing equation can be written in its 2-dimensional form. Motz and Klanfer<sup>14</sup> have used relaxation to calculate the field distribution in an electrostatic lens, consisting of two semi-infinite cylinders placed end to end and separated by a finite gap. Hesse<sup>15</sup> recently applied the method to the calculation of the field distribution inside a magnetic electron lens, taking account of saturation of the pole pieces.

#### Solution of Special Partial Differential Equations

##### TYPES OF EQUATIONS

Difference equations can be formed to replace most differential equations occurring in electrical engineering, and the appropriate relaxation patterns found by similar methods to those already described. If, however, the equation is very complicated, so is the difference equation, and the relaxation pattern may involve a great number of points. It is still possible to work with such patterns as Southwell<sup>16</sup> has shown, but the manipulation then becomes rather involved and tedious. In these cases, however, solution by any other method is practically impossible and so the labor involved in relaxation is justified.

##### EQUATIONS CONTAINING MIXED DERIVATIVES

The finite difference approximations to mixed derivatives can be found quite easily. With the arrangement of points as shown in Figure 30, a Taylor expansion

\* The author wishes to thank Dr. H. Motz and the Institution of Electrical Engineers, London, England, for permission to reproduce Figure 29 from one of his papers.

leads to this approximation

$$4a^2 \left( \frac{\partial^2 V}{\partial x \partial y} \right)_0 - V_A - V_B + V_C - V_D$$

The use of this can best be seen from this example

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + 2 \frac{\partial^2 V}{\partial x \partial y} = 0$$

This can be written approximately

$$\frac{V_1 + V_3 - 2V_0}{a^2} + \frac{V_2 + V_4 - 2V_0}{a^2} + 2 \frac{V_A - V_B + V_C - V_D}{4a^2} = 0$$

whence the residual equation is

$$R_0 = \Sigma V_1 - 4V_0 + \frac{1}{2}(V_A - V_B + V_C - V_D) \quad (12)$$

The relaxation pattern corresponding to equation 12 is shown in Figure 31.

#### QUASI-PLANE POTENTIAL EQUATION

This is the name given to the equation

$$\frac{\partial}{\partial x} \left( \epsilon \frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial y} \left( \epsilon \frac{\partial V}{\partial y} \right) + Z = 0$$

which reduces to Poisson's equation, (that is, the plane harmonic equation) when  $\epsilon$  is constant. It is, for example, the equation for the electric field  $V$  in a region of nonuniform dielectric constant  $\epsilon$ .

With the arrangement of points as in Figure 32, and denoting the value of  $\epsilon$  between  $P_0$  and  $P_1$  by  $\epsilon_1$ , and so forth, the finite difference approximation to this equation is found to be

$$\frac{1}{a^2} [\epsilon_1(V_1 - V_0) - \epsilon_3(V_0 - V_3) + \epsilon_2(V_2 - V_0) - \epsilon_4(V_0 - V_4)] + Z = 0$$

whence the residual formula 13 is obtained

$$R_0 = \sum_{n=1}^4 \epsilon_n V_n - U_0 \sum_{m=1}^4 \epsilon_m + a^2 Z \quad (13)$$

The corresponding relaxation pattern is shown in Figure 33.

#### RESONATOR AND OSCILLATION PROBLEMS

In most oscillation problems (for example, the field distribution inside a resonator) the governing equation is of the form

$$\nabla^2 V + k^2 V = 0 \quad (14)$$

The boundary conditions in waveguide and resonator problems are such that this equation has nontrivial solutions only for certain discrete values of  $k^2$ . Solving the equations therefore involves a

determination of the allowed parameters  $k^2$  (the "characteristic values") as well as of the corresponding function  $V$ . They can be solved by a combination of relaxation and iteration, making use of Rayleigh's principle which gives for the fundamental mode

$$k^2 = - \frac{\int \int V \cdot \nabla^2 V dx dy}{\int \int V^2 dx dy} \quad (15)$$

The procedure is to assume values for  $V$  and with these values calculate  $k^2$  from equation 15. This value is substituted in equation 14 which then can be relaxed in the usual way. From this a corrected value of  $V$  is found which is used in equation 15 to obtain a corrected value of  $k^2$ , and so forth. The process is continued until corrections are negligible, 3 or 4 cycles usually being sufficient.

Allen, Fox, Motz and Southwell<sup>17</sup> applied this treatment to a 2-dimensional section through a klystron tube. Motz<sup>18</sup> extended the method to finding the field distribution and resonance frequency of cavity resonators, improving the accuracy by taking account of the analytic behaviour of fields in the region of sharp corners.

#### Conclusion

Southwell and his co-workers, who developed the method of relaxation, were interested mainly in applying it to problems in mechanical engineering. However, as this account has shown, the principles and techniques established by them are easily applicable to many problems in electrical engineering. The most important and most frequent use of the method is in the solution of the electromagnetic field problems described by Laplace's, Poisson's, and Maxwell's equations. Its great advantages as compared with analytical methods are:

1. It is simple to apply.
2. The process is essentially the same for all types of equations.
3. It can be checked easily during the progress of the work.
4. It can be applied to problems for which no analytical solution can be obtained.
5. It can mostly be made as accurate as a given problem requires it to be.

The chief disadvantage is that the solution obtained is a numerical one and applies only to the one case for which it has been worked. A variation of any parameter necessitates a complete recalculation. The solution for the first case, however, often gives some guidance in the solution of the next, thus reducing

the effort required for the whole range of solutions.

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No Discussion

# Notes on the Design of Eccles-Jordan Flip-Flops

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**Synopsis:** The design of a prototype Eccles-Jordan flip-flop such as used in large-scale digital computers is discussed, with particular attention to the role played by component variability. A graphical design technique is described; the technique should prove useful in other switching-circuit design problems where a large number of identical direct-coupled circuits are employed.

A preliminary analysis is carried through for the rise time of the on-going control grid. The results may be used to indicate the influence of transpose capacitance on speed of switching of flip-flops.

IT IS common knowledge that the bistable vacuum-tube device referred to as the Eccles-Jordan flip-flop was first described by Eccles and Jordan in 1919.<sup>1</sup> It is perhaps not so well known that the device described by them is not identical with the current flip-flop in that they used batteries in the place of transpose resistors and thus dispensed with resistor divider networks.

The flip-flop now in common use is shown in Figure 1. It is seen to consist of the following: two divider networks each containing a plate resistor  $R_1$ , a transpose resistor  $R_2$  usually shunted by a transpose capacitor  $C_2$ , and a grid resistor  $R_3$ ; two tubes, either triodes or pentodes; and two supply voltages, one positive and the other negative with respect to the grounded cathodes. Variations of this configuration are in use; note in particular that it is not necessary that the two halves of the flip-flop be nominally identical. However, this paper will restrict consideration to the illustrated circuit.

Figure 1 illustrates one stable state in which the left tube is completely cut off while the grid voltage of the right tube is slightly above ground. A. T. Starr<sup>2</sup> showed that stability can be attained even though the "off" tube is not cut off and the "conducting" tube has negative grid. In fact, it is easy to show that stability requires only that, in the quiescent state, the loop gain around the two tubes be less than unity. On the other hand, it is advisable to protect against malfunctioning due to noise and, therefore, to reduce the loop gain by ensuring both cutoff for one tube and grid conduction for the

other. In any case, it should be noted that a variation in any parameter which increases the grid voltage of the cutoff tube or decreases the grid voltage (or current) of the conducting tube leads the flip-flop closer to instability; this criterion will be used later to determine the influence of parameter variations on flip-flop stability.

## Scope of Paper

The design of a single Eccles-Jordan flip-flop for laboratory use is relatively simple and straightforward. On the contrary, the design of a prototype flip-flop for use, for example, in a large-scale digital computer requires more care because of the large number of flip-flops involved. First, the flip-flop must fit into the framework set by other (computer) circuits, which implies specified output voltage levels and switching times. Secondly, the flip-flop must exhibit the desired characteristics in spite of component variations, both initial and during use in the machine. In this respect, attention must be focussed on resistor drift, tube aging, and fluctuations in supply voltages.

The general problem of flip-flop design with all its ramifications is much too complex to permit of its discussion here. Hence, the following assumptions are made in order to restrict and to define the scope of the paper:

1. All resistors will be so selected that after long service in the computer they will not drift outside a given fractional tolerance  $\rho$  from nominal resistance.
2. All supply voltages can be held at all times to within a given fractional tolerance  $\sigma$ .
3. Knowledge of tube characteristics for the complete life of every tube (or tube rejection criteria) are such that: 1. plate current  $i_p$  for zero grid voltage  $e_g$  and specified plate voltage  $e_p$  will always exceed a known value, and 2. grid bias for given  $e_p$  need never exceed a known value to ensure cutoff.
4. Satisfactory operation is ensured if: 1. tube cutoff is ensured, 2. the grid of the conducting tube reaches at least zero bias, 3. zero grid bias implies zero grid current. (The grid current condition has been included only to simplify exposition. The

graphical design procedure described in this paper is readily modified to take account of grid current at zero grid bias.)

5. Only the grounded-cathode flip-flop is to be considered, and no inductors are to be used.

## Notation

The notation used in this paper is illustrated in Figure 1.  $E_1$  and  $E_2$  refer to the positive and negative supply voltages, while  $E$  and  $G$  refer to plate voltage and grid voltage respectively. Algebraic signs have been appended so that all symbols are positive quantities. The subscripts  $C$  and  $NC$  refer to the  $C$  and  $NC$  divider networks connected to the conducting and nonconducting anodes respectively. Finally, the transpose capacitance  $C_2$  is assumed to be a variable parameter available to the designer, but the stray capacitances  $C_1$  and  $C_3$  in shunt with  $R_1$  and  $R_3$  respectively are fixed by tube type and physical layout. In this regard it is worth mentioning that  $C_3$  does not include Miller capacitance; this is certainly justifiable for the off tube, which is the only case treated here, but can be largely justified in any case.

## Derivation of the "Worst" Configuration

It is proposed now to determine the "worst" possible configuration of the parameters, that is, that configuration most nearly approaching instability as defined above. A graphical approach is to be employed, partly because it has proved to be simpler than the analytic method, but mainly because the graphical method is most illuminating and immediately applicable to design of more-sophisticated flip-flops, of univibrators, and of all circuits using resistance divider networks.

Consider first the  $NC$  divider illus-

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The author would like to point out that the work reported here was begun at the Institute for Advanced Study Computer Laboratory and to acknowledge incorporation of some of the basic concepts prevalent there. Credit belongs to Mr. P. Levonian for checking some of the lemmas experimentally, to Mr. S. Ruhman for verifying the algebraic steps leading up to the lemmas, and to the clerical staff of the Moore School for their greatly appreciated assistance.

The latter and greater part of the work described here has been done in connection with Signal Corps Contract Number DA-36-039-sc-14 for Electronic Computer Research and Development, placed by Signal Corps Engineering Laboratories, Department of the Army, Fort Monmouth, N. J.

trated in Figure 2. Note in the upper graph the "minimum" plate characteristic for  $e_g = 0$ , item 3 in the section "Scope of Paper." The divider current  $i_{NC}$  is drawn as if it were negative plate current, the reason to be given later. The solid lines labelled  $R_1$ ,  $R_2$ , and  $R_3$  are seen to be "load lines" for the respective resistors, that is, their slopes are the reciprocals of the resistances represented. The broken lines show the effect of more negative  $-E_2$  supply (larger  $E_2$ ). Clearly, the resultant change in  $G$  tends toward flip-flop instability since the originally zero grid goes negative.

Similarly, the lower graph of Figure 2 shows that an increase in  $R_2$  in the NC divider tends toward instability. By graphically studying the effect of varying each parameter, one at a time, its influence on stability is rapidly ascertained.

The C divider can be studied graphically in similar fashion, as shown in Figure 3. Here again the solid lines are load lines. Note in particular that the load line for  $R_1$  fits properly onto this graph only because the divider current  $i_c$  has been treated as negative  $i_p$ . The graph shows that decreasing  $R_1$  tends to instability because it tends to raise the grid voltage of the nonconducting tube. Actually, it is possible to derive more information from this graph by noting that the change  $\Delta G$  is due to increased  $i_c$  which is in turn due to increase in  $E_c$ . It follows that decreasing  $R_1$  will tend to instability whenever the plate characteristic has a positive finite slope at the point  $(E_c, i_p)$ , a condition generally realized.

The effect of variation of each parameter of the C divider can be studied graphically to yield information regarding tendency toward instability as well as associated conditions on the minimum plate characteristic. Although this is not

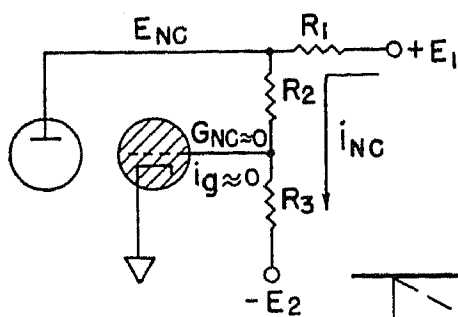
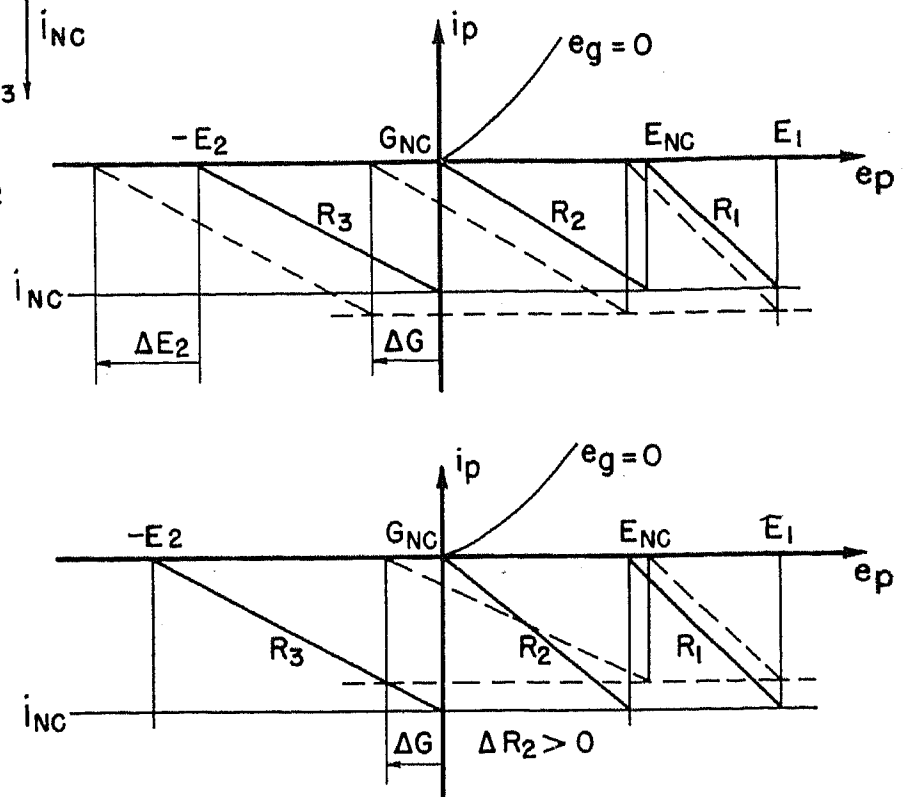


Figure 2. Influence of NC-divider parameters on stability



to be carried through in this paper, the results can be summarized as follows:

*Lemma I:*

(a). For the NC divider, the most nearly unstable configuration is obtained with  $E_1(\min)$ ,  $R_1(\max)$ ,  $R_2(\max)$ ,  $R_3(\min)$ , and  $E_2(\max)$ :

(b). For the C divider, the most nearly unstable configuration is obtained with  $E_1(\max)$ ,  $R_1(\min)$ ,  $R_2(\min)$ ,  $R_3(\max)$ , and  $E_2(\min)$ .

Study of the configurations of lemma I will show that they do not result in extreme values of  $E$ ; nevertheless, the notation  $E(\max)$ ,  $E(\min)$  will be used for this case.

One might object that I(a) and (b) are inconsistent since  $E_1$  and  $E_2$  are the same

for both dividers and cannot be both maximum and minimum simultaneously. However, the purpose is to design a flip-flop stable over long periods of time. Hence, it is necessary to apply both parts of the lemma simultaneously in the initial design. Each part was independently derived for the same reason.

### Some Design Lemmas

Lemma I can be applied to derive analytic expressions for the network resistances. Thus from I(a) it follows that

$$i_{NC} = \frac{E_1(\min)}{R_1(\max) + R_2(\max)} = \frac{E_2(\max)}{R_3(\min)} \quad (1)$$

while from I(b) is obtained

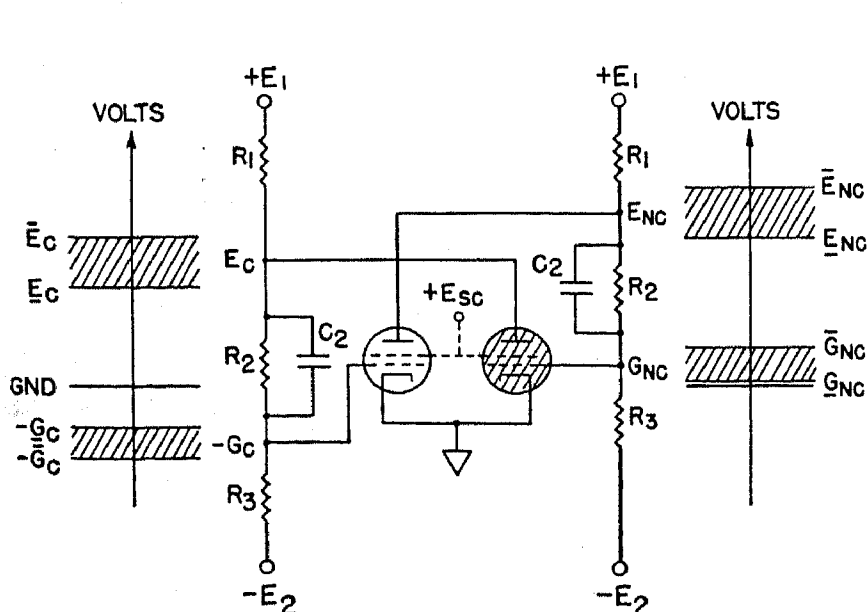


Figure 1. Grounded-cathode Eccles-Jordan flip-flop. (Supercript and subscript bars denote maximum and minimum respectively)

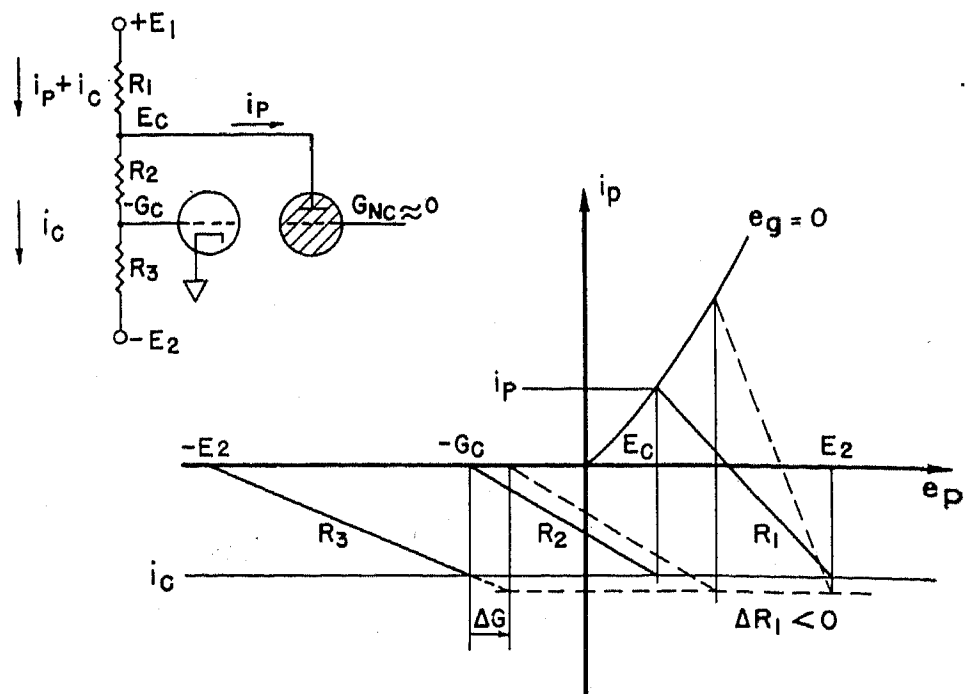


Figure 3. Influence of C-divider parameters on stability

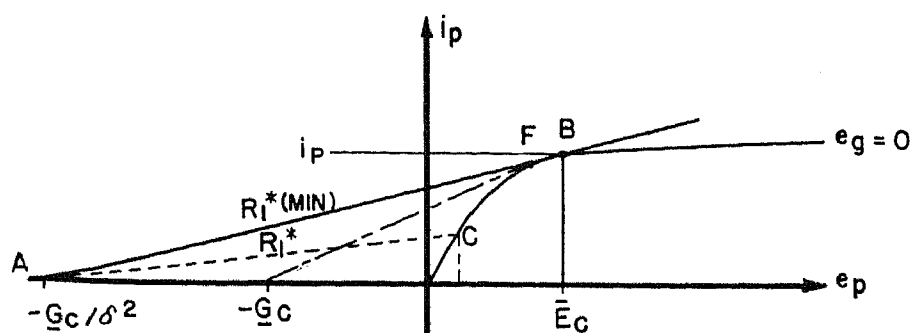


Figure 4 (left). Determination of optimum  $E_c(\max)$ . (Superscript and subscript bars denote maximum and minimum respectively)

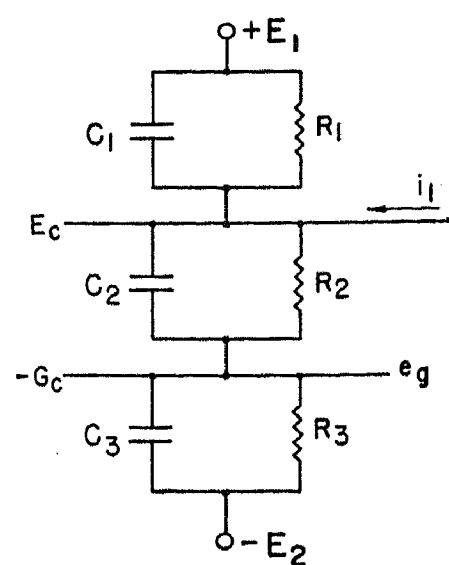
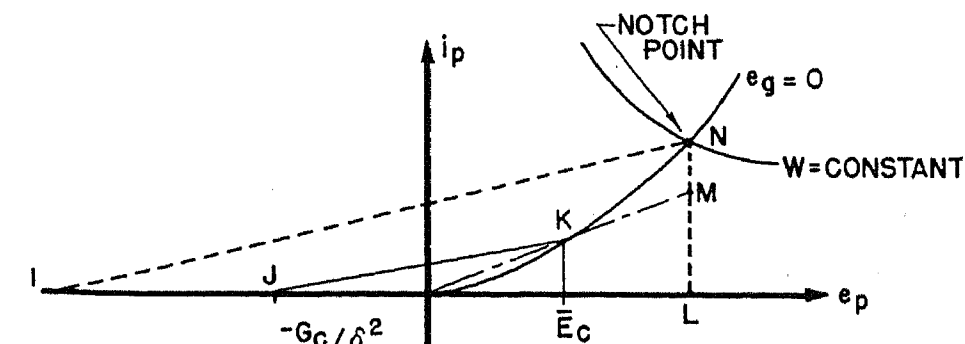


Figure 6 (right). Initial conditions for rising grid voltage



$$i_c = \frac{E_c(\max) + G_c(\min)}{R_2(\min)} = \frac{E_2(\min) - G_c(\min)}{R_3(\max)} \quad (2)$$

$$i_p + i_c = \frac{E_1(\max) - E_c(\max)}{R_1(\min)} \quad (3)$$

These equations can be transferred by straightforward algebraic manipulation to

$$i_p R_1(\min) = E_1(\max) \delta^2 + G \quad (4)$$

$$i_p R_2(\min) = \frac{E + G}{1 - g} \times \left[ \frac{1}{\beta^2 - \frac{E + G}{(E_1(\max) - E)(1 - g)}} - (1 - g) \right] \quad (5)$$

$$i_p R_3(\max) = E_2(\min) \times$$

$$\left[ \frac{1}{\beta^2 - \frac{E + G}{(E_1(\max) - E)(1 - g)}} - (1 - g) \right] \quad (6)$$

$$\beta^2 = \frac{E_1(\max)}{E_1(\max) - E} \theta^2 \quad \delta^2 = 1 - \theta^2(1 - g)$$

$$\theta = \frac{(1 - \rho)(1 - \sigma)}{(1 + \rho)(1 + \sigma)} \quad g = \frac{G}{E_2(\min)}$$

$$E \equiv E_c(\max) \quad G \equiv G_c(\min)$$

The following lemmas may be seen to follow almost immediately from equations 4, 5, and 6:

**Lemma II:**

$R_1$ ,  $R_2$ , and  $R_3$  are all inversely proportional to  $i_p$ ; that is, all other things equal, the resistances are minimized by choosing a tube with maximum perveance. (This lemma is modified slightly when  $i_q \neq 0$  for  $e_g = 0$ .)

**Lemma III:**

$R_1$  is increased,  $R_2$  and  $R_3$  are decreased

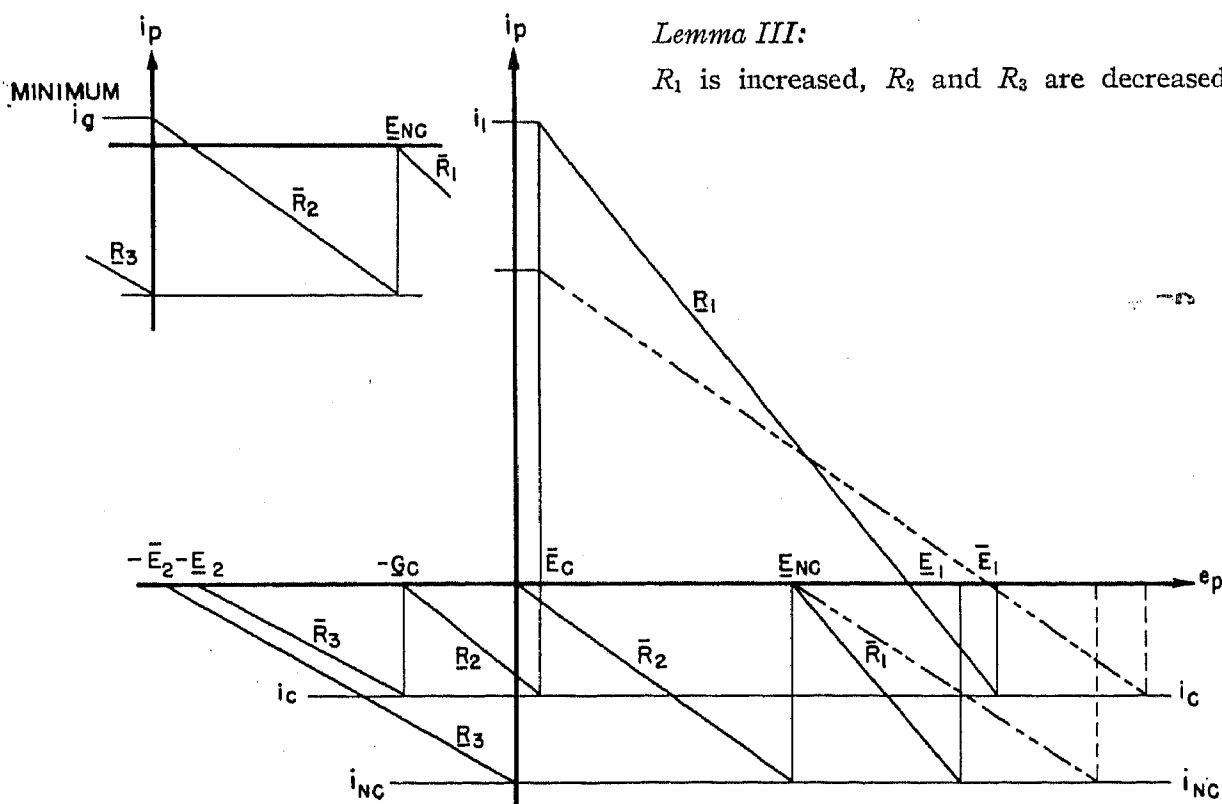


Figure 5. Graphical flip-flop design. (Superscript and subscript bars denote maximum and minimum respectively)

when a larger value of  $E_1$  is chosen.

**Lemma IV:**

If  $E_2$  be chosen appreciably larger than  $G$  so that  $g$  is small, then  $R_1$  and  $R_2$  are almost independent of  $E_2$ .  $R_3$  is approximately proportional to  $E_2$ .

**Lemma IV(a):**

A reasonable choice for  $E_2$  is  $E_2 \approx 10G_c(\min)$ .

**Lemma V:**

All three resistors increased as either resistance tolerance  $\rho$  increases or supply tolerance  $\sigma$  increases or both, for then  $\theta$  and  $\beta$  decrease and  $\delta$  increases.

Since neither  $R_2$  nor  $R_3$  can be negative, it is necessary for the denominator of the first term inside the bracket of equations 5 and 6 to be positive. That is

$$\beta^2 - \frac{E + G}{(E_1(\max) - E)(1 - g)} > 0 \quad (7)$$

and therefore

**Lemma VI:**

The minimum permissible value of  $E_1$  is limited by the inequality

$$E_1(\max) > \frac{E + G}{1 - g} \cdot \frac{1}{\theta^2}$$

Finally it is proposed to derive a criterion for choice of  $E_c(\max)$ . The criterion will be based on the assumption, believed to be approximately true, that flip-flop speed is inversely proportional to  $R_1$ , that is, that  $R_1$  should be minimized. By lemma III, this occurs when  $E_1$  is chosen as small as possible. Applying lemma VI to equation 4, one obtains

$$i_p R_1(\min) > \frac{(E + G)\delta^2}{(1 - g)\theta^2} + G = \frac{\delta^2}{1 - \delta^2} \left( E + \frac{G}{\delta^2} \right) \quad (8)$$

It is helpful to introduce the resistance  $R_1^*$  given by

$$i_p R_1^* = E + \frac{G}{\delta^2} \quad (9)$$

Thus,  $R_1^*$  is a convenient measure of  $R_1$ .

Consider first a flip-flop designed around a pair of pentodes so that  $G$  is fixed by the screen voltage. The upper graph of Figure 4 illustrates a construction for obtaining  $R_1^*$  as defined by equation 9. The procedure consists of drawing a line such as  $AC$  from the point  $(-G/(\delta^2), 0)$  to intersect the minimum plate characteristic at  $C$ ; if  $E$  be chosen to equal the plate voltage at  $C$ , the slope of the line  $AC$  is inversely proportional to  $R_1^*$ . Clearly,  $R_1^*$  is minimized when the line is drawn tangent to the characteristic, as indicated by the solid line  $AB$  (labelled  $R_1^*(\min)$ ). The point of tangency determines the optimum value of  $E$ . Since  $0 < \delta < 1$ ,  $R_1^*$  can never be decreased below the value corresponding to the dashed line passing through  $(-G_c(\min), 0)$  and tangent at  $F$ . Clearly,  $E$  should be chosen somewhere on the "knee" of the curve.

In the case of a triode flip-flop, the same reasoning leads to the result that  $E$  be chosen as large as possible. The upper limit is set by the rated anode power dissipation  $W$ . Hence,  $E$  should be chosen at the "notch point"  $N$  at which the minimum plate characteristic intersects the "rated power" curve  $W = \text{constant}$  as shown in the lower graph of Figure 4. The broken line  $IN$  indicates the corresponding  $R_1^*(\min)$ .

It should be recalled, however, that in the case of a triode,  $G$  is roughly proportional to  $E_1$  which is in turn roughly proportional to  $E$ . Hence  $R_1^*$  does not increase with decreasing  $E$  as rapidly as might otherwise be anticipated. The solid line  $JK$  shows the effect of reducing all voltages by a factor of two. By drawing a line from the origin through the new

design point  $K$  to intersect  $LN$  at  $M$  and by applying the lemmas of similar triangles, it can be shown that the new design point results in an increase in  $R_1^*$  in the ratio of  $LN$  to  $LM$ . In practice, the design point must be chosen well below the notch point because of other considerations, such as resistor power dissipation and magnitude of supply voltages. Consequently, this construction is a useful guide to the resultant increase in  $R_1$ .

The preceding comments may be summarized as follows:

*Lemma VII:*

- (a) For a pentode flip-flop,  $E_c(\max)$  should be chosen on the knee of the minimum plate characteristic.
- (b) For a triode flip-flop,  $E_c(\max)$  should be chosen as large as other considerations permit.

### Graphical Design Procedure

The analytic expressions for the network resistances given by equations 4, 5, and 6 of the section entitled "Some Design Lemmas" contain the following seven parameters,  $E_1$ ,  $E_2$ ,  $E$ ,  $G$ ,  $i_p$ ,  $\rho$ ,  $\sigma$ . The magnitudes of these parameters may be assigned arbitrarily by the designer, subject only to the condition on  $E_1$  stated in lemma VI. The assignment of values for  $\rho$  and  $\sigma$  is dependent upon knowledge of the characteristics of the resistors and of the power supplies to be used, while the selection of  $G$  and  $i_p$  depends upon a knowledge of the characteristics of existing vacuum tube types; clearly, engineering judgment must enter into the selection of values for these. At the same time, lemmas IV, V and VII serve as use-

ful guides which restrict the ranges of  $\rho$ ,  $\sigma$ ,  $E_2$  and  $E$ .

The remarks of the preceding paragraph suggest that all seven parameters may be chosen in advance and that the analytic expressions of equations 4, 5, and 6 may be used directly for determining the values of  $R_1$ ,  $R_2$ , and  $R_3$ . Where flip-flop application is not overly restricted by side conditions, this is probably true. However, it has been the author's experience in computer design that side conditions force the engineer to try many possible flip-flop designs. In this event the computation of the analytic expressions is time consuming and the graphical approach now to be described is much faster. Moreover, as we have seen, the graphical method immediately illustrates the influence of varying each parameter.

The design procedure is based directly upon lemma I and the assumptions specified in the section entitled "Scope of the Paper." The procedure is illustrated in Figure 5.

1. Set up  $i_p$ ,  $e_p$  axes; the voltage scale is known but the current scale is unknown initially.
2. Draw a line parallel to and below the  $e_p$  axis to represent  $i_c$ , the  $C$ -divider current. (It is convenient to make  $i_c = 10$  divisions.)
3. Locate  $-E_2(\min)$ ,  $-G_c(\min)$ ,  $E_c(\max)$ . Optimum choices for  $E_2$  and  $E$  are based on lemmas IV(a) and VII;  $-G_c(\min)$  is the guaranteed grid cutoff bias. Draw load lines for  $R_2(\min)$  and  $R_3(\max)$ .
4. Draw the load line for  $R_3(\min)$  in the  $NC$  divider. This line passes through  $(-E_2(\max), 0)$  and  $(-G_c(\min) - [E_2(\max) - E_2(\min)], i_c[1 + \rho/1 - \rho])$ . Produce this line to cut the current axis; the intersection is at  $i_{NC}$ , the current through all three  $NC$ -divider resistors.
5. Draw the load line for  $R_2(\max)$  in the  $NC$  divider; it passes through  $(0, 0)$  and  $[E_c(\max) + G_c(\min), i_c(1 - \rho/1 + \rho)]$ . Produce the line to  $i_{NC}$ ; the corresponding voltage is  $E_{NC}$ .
6. Draw the load line for  $R_1(\max)$  in the  $NC$  divider; it passes through  $(E_{NC}(\min), 0)$  and  $(E_1(\min), i_{NC})$ .
7. Draw the load line for  $R_1(\min)$  in the  $C$  divider; it passes through  $(E_1(\max), i_c)$  and  $[E_{NC}(\min) + E_1(\max) - E_1(\min), i_c - i_{NC}(1 + \rho/1 - \rho)]$ . Produce to intersect  $E_c(\max)$  at  $i_p$ . Then  $i_p$  is the minimum tube current at  $e_p = E_c(\max)$  for the most nearly unstable configuration, that is it is the current obtained from the assumed minimum plate characteristic. Hence this determines the current scale for  $i_{NC}$  and  $i_c$ . Consequently all resistance values can now be obtained from this graph.

Since this procedure has been based on lemma I, an analysis must show that any variation of one or more parameters within the allowed limits results in a stable flip-flop configuration.

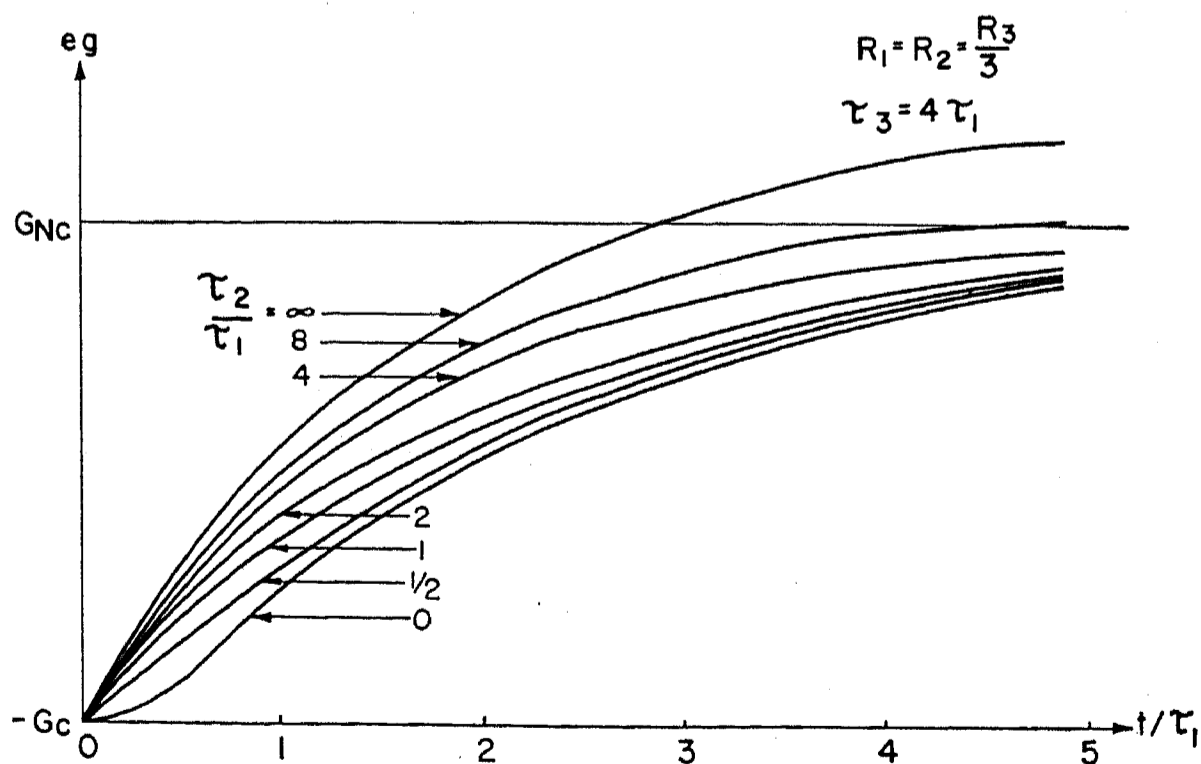


Figure 7. The influence of  $\tau_2$  on switching time

No matter how the resistance values are calculated, it will in general happen that non-Radio Manufacturers Association values will result. Lemmas II, III, and IV may be applied to rectify this situation. Thus, since  $i_p$  has been determined using engineering judgment, it is quite reasonable to alter the preassumed value by some small percentage; this may be used to effect a small proportional rise or fall in all resistances. Again, a small percentage increase or decrease in  $E_1$  may be used to vary the resistances in a nonproportional manner. Finally,  $R_3$  may be adjusted with negligible effect on  $R_1$  and  $R_2$  by varying  $E_2$ . (Where these lemmas can not be used, as for example when  $E_1$  and  $E_2$  cannot be altered, the non-Radio Manufacturers Association difficulty may be alleviated by designing with larger tolerances,  $\rho$  and  $\sigma$ , than engineering judgment demands.)

The power of the graphical method is illustrated by two remarks. First, the broken lines on the right-hand side of Figure 5 instantly depict the influence of increased  $E_1$  on the divider resistors. Since  $i_p$  is given and does not change, larger  $E_1$  results in larger  $i_{NC}$  and  $i_c$  and hence in smaller  $R_2$  and  $R_3$ ; at the same time,  $R_1$  increases as is evident from inspection of the intersection of the two load lines for  $R_1(\min)$  with the voltage axis. Second, the graphical approach is easily extended to take account of grid current at zero grid volts; the method is indicated by the insert top left of Figure 5. The author has used the graphical method for a case when the analytic expressions are particularly complex; this was the design of a flip-flop with a guaranteed minimum grid current and also the ability to drive an NE-17, connected to the anode, with a prescribed minimum current. The saving in design time was deemed to be substantial.

## General Switching Considerations

There are at least two switching times of importance which must be distinguished. The first is the repetition rate that is, the rate at which the flip-flop can switch repeatedly from one state to the other and back. Repetition rate is related to the settling time of the flip-flop, that is, the time required for the flip-flop to reach its new quiescent state.

The second switching time of importance is the switching speed, that is, the speed at which an initially quiescent flip-flop changes state. The switching speed is intimately related to the minimum duration of trigger pulse for successful switching. It will be shown that in general

switching speed and repetition rate cannot be maximized simultaneously.

In both cases, the time of arrival at the new state must be defined *a priori*, say, through specification of output voltage.

Switching time is necessarily dependent on the kind of triggering used and on the amount of energy provided by the trigger pulse. For example, the ultimate in speed is achieved by a trigger which, by direct application to all appropriate points, instantaneously switches all voltages to their new values.

In practice, however, the trigger is applied to only one electrode for a time sufficient to insure change of state. It is proposed to consider the case when the conducting grid is instantaneously cut off. The problem is then reduced to determining the rate of rise of grid voltage of the initially nonconducting tube in response to a step of current,  $i_1$ , injected into the  $C$  divider between  $R_1$  and  $R_2$ , as indicated in Figure 6. It is easily shown that

$$e_g(t) = G_{NC} - (G_{NC} + G_C) \times \frac{1}{[(1+b)e^{-\alpha t} - be^{-\gamma t}]} \quad (10)$$

where

$$b = b(\tau_2) = \frac{\alpha(1 - \gamma\tau_2)}{(\gamma - \alpha)}$$

$$\gamma, \alpha = \frac{1}{2R_1\tau_j\tau_k} [(R_i + R_j)\tau_k \pm S]$$

$$S^2 = [(R_i + R_j)\tau_k]^2 - 4(R_i)(R_j\tau_k)$$

$$\tau_k = R_k C_k$$

$$R_i = R_1 + R_2 + R_3$$

$$R_j\tau_j\tau_k = R_1\tau_2\tau_3 + R_2\tau_3\tau_1 + R_3\tau_1\tau_2$$

$$(R_i + R_j)\tau_k = (R_1 + R_2)\tau_3 + (R_2 + R_3)\tau_1 + (R_3 + R_1)\tau_2$$

Also it can be shown that  $S^2 \geq 0$  and therefore  $\gamma \geq \alpha$ . The equality signs hold if and only if  $\tau_1 = \tau_2 = \tau_3$ .

In what follows, it is assumed that  $\tau_2$  is the only explicit variable available to the designer and that all other terms are parameters adjustable within the limits set by stability requirements. An inspection of equation 10 shows that the transient term consists of two exponential terms, of which the first requires longer to decay since  $\alpha < \gamma$ . Moreover, the variation of  $\gamma$  and  $\alpha$  with  $\tau_2$  can be obtained immediately from the expressions of equation 10. Thus it can be shown

**Lemma VIII:**

- (a)  $\gamma \geq H$ ;  $\gamma \rightarrow H$  as  $\tau_2 \rightarrow \infty$ , but limit  $\gamma \geq H$
- (b)  $\alpha \leq H$ ; limit  $\alpha = 0$  as  $\tau_2 \rightarrow \infty$

where

$$H = \frac{(R_1 + R_2 + R_3)(R_1\tau_3 + R_3\tau_1)}{(R_1\tau_3 + R_3\tau_1)^2 + R_1R_2\tau_3^2 + R_2R_3\tau_1^2}$$

When  $\tau_1 = \tau_3$ , then  $H = 1/\tau_1$  and the

equality signs in lemma VIII are obeyed. Moreover, note that the largest value of  $\alpha$  is less than the smallest value of  $\gamma$ , regardless of the value of  $\tau_2$ , for any specific design.

The coefficient  $b$  has been defined in equation 10; in particular, it can be seen that

$$b(0) = \frac{\alpha}{\gamma - \alpha} > 0 \quad (11)$$

It follows that both coefficients are positive for  $\tau_2 = 0$ . It can be shown that:

**Lemma IX:**

As  $\tau_2$  increases, both coefficients decrease through zero and become negative. In particular,  $b = 0$  when  $\tau_2$  equals the smaller of  $\tau_1$  and  $\tau_3$ ;  $(1+b)$  equals zero when  $\tau_2$  equals the larger of  $\tau_1$  and  $\tau_3$ .

Since  $\alpha$  can never exceed  $\gamma$ , minimum settling time can be achieved only by eliminating the term exponential  $(-\alpha t)$ , that is, by making  $(1+b) = 0$ . Thus

**Lemma X:**

To achieve minimum settling time,  $\tau_2$  should be chosen equal to the larger of  $\tau_1$  and  $\tau_3$ . Settling time is then dependent only upon  $\gamma$ .

Two observations are worthy of note. First, this value of  $\gamma$  does not differ too markedly from  $1/\tau_1$ . It follows that the magnitude of  $\tau_1$  constitutes a good first approximation to flip-flop speed. This circumstance was utilized in the section entitled "Graphical Design Procedure" in choosing optimum  $E_c(\max)$  on the basis of minimum  $R_1$ . Second, when  $\tau_2$  equals the larger of  $\tau_1$  and  $\tau_3$ , the equation for  $\gamma$  simplifies to

$$\gamma = \frac{R_1 + R_2 + R_3}{(R_1 + R_2)\tau_3 + R_3\tau_1}; \quad \tau_1 = \tau_2 > \tau_3 \quad (12)$$

$$\gamma = \frac{R_1 + R_2 + R_3}{R_1\tau_3 + (R_2 + R_3)\tau_1}; \quad \tau_1 < \tau_2 = \tau_3$$

As pointed out, equation 12 was chosen to satisfy minimum settling time. It is common experience, however, that repetition rate can be improved by introducing a small amount of overshoot at the expense of settling time. It is not difficult to show the following:

**Lemma XI:**

Overshoot occurs if and only if  $\tau_2$  is greater than both  $\tau_1$  and  $\tau_3$ .

It has been found experimentally for one flip-flop design that maximum repetition rate was obtained when  $\tau_2$  exceeded the larger of  $\tau_1$  and  $\tau_3$  by about 10 per cent.

To obtain maximum switching speed requires that the grid voltage  $e_g$  rise at its maximum rate. By differentiating  $e_g$  with respect to time  $t$  in equation 10, it can be shown that:

### Lemma XII:

For maximum switching speed,  $\tau_2$  should be so chosen that

$$\tau_2 > \frac{R_2\tau_3\tau_1}{R_1\tau_3 + R_3\tau_1} \text{ or } C_2 > \frac{C_1C_3}{C_1 + C_3}$$

In words,  $C_2$  should be chosen much larger than the capacitance of the series connection of  $C_1$  and  $C_3$ .

The influence of  $\tau_2$  on switching speed is illustrated by the curves of Figure 7, which show  $e_g$  versus time for various values of  $\tau_2$ . Note that these curves are pictorial verification of lemmas X, XI, and XII.

The use of large  $\tau_2$  to obtain maximum switching speed implies the existence of appreciable overshoot. In the case of the rising grid, this is of no consequence since the grid is clamped at cathode voltage. On the other hand, it is the same network which is involved in fall time considerations when the flip-flop returns to its

original state. Consequently, there will be a similar overshoot in the case of the falling grid, probably of increased magnitude because of the initial grid clamping. The overshoot reduces the ultimate repetition rate since it causes the grid, when retriggered soon after having been triggered, to start out from a voltage below  $-G_c$  and hence to take longer to rise. Thus it appears qualitatively that the two switching times, repetition rate and switching speed, cannot be optimized simultaneously.

### Conclusion

This paper has been concerned with the design of Eccles-Jordan flip-flops. After restricting the scope of the discussion to a particular design and to specific assumptions, analytic design expressions were obtained and a number of lemmas developed with regard to optimizing param-

eters while guaranteeing stability within stated parameter tolerances. Finally, a general graphical design procedure was developed.

The use of a transpose capacitor to improve switching times was discussed and a number of lemmas developed to serve as a guide not only in the choice of transpose condenser but also in choice of other parameters (particularly  $R_1$  and  $R_3$ ) within the stability limits previously obtained.

The paper makes no claim to completeness but is meant only to serve as a collection of rules to guide the design of flip-flops.

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## No Discussion

# Ultrathin Tapes of Magnetic Alloys with Rectangular Hysteresis Loops

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**Synopsis:** The principal soft magnetic alloys have been rolled to thicknesses between 1/8 and 1 mil. Magnetic properties of toroidal cores wound from ultrathin tapes were studied using d-c magnetization. Even though the coercive force tends to increase with reduced thickness, the coercivity of ultrathin metal tapes is usually lower than that of ferrites or powdered metals. Relatively rectangular d-c hysteresis loops were obtained in three of four alloys investigated.

**T**HE continuing need for magnetic core materials which can be used effectively at high frequencies has stimulated the search for materials in which eddy-current effects produced by rapid changes in magnetic flux are at a mini-

mum. Generally speaking there are two ways in which eddy-current effects can be reduced: increasing the electrical resistivity as in the ferrites, or restricting the paths in which eddy-currents can flow, as

in powdered metals or thin laminations of metal strip.

The ferrites, which have resistivities about  $10^8$  to  $10^{12}$  higher than conventional magnetic alloys, have been used extensively in television sweep transformers where they may be operated at high frequency with low power loss.<sup>1</sup> However, such characteristics as relatively low magnetic saturation, low Curie point and certain dielectric properties, particularly for larger masses, restrict the scope of application of the ferrites. On the other hand, the fine subdivision of particles in powdered metal

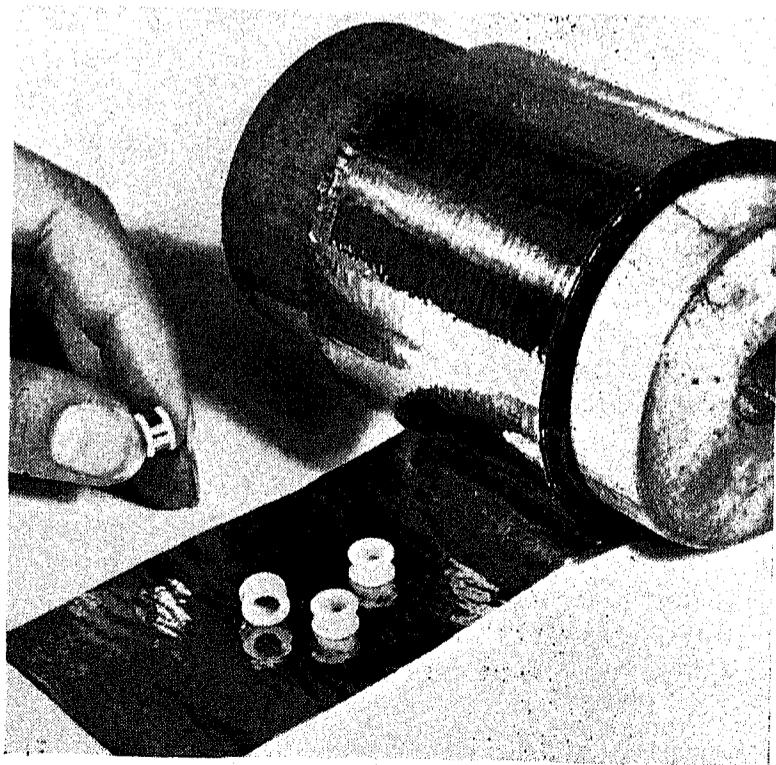


Figure 1. Coil of cold rolled 48-per-cent nickel-iron alloy 1/4-mil thick and ceramic bobbins

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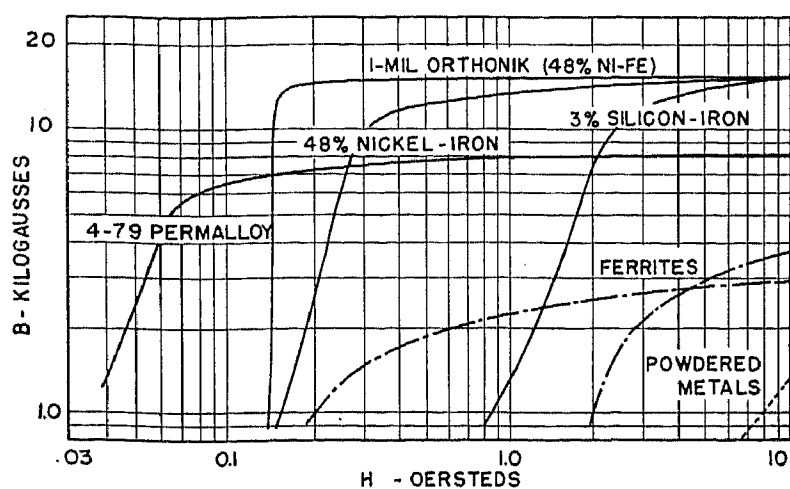
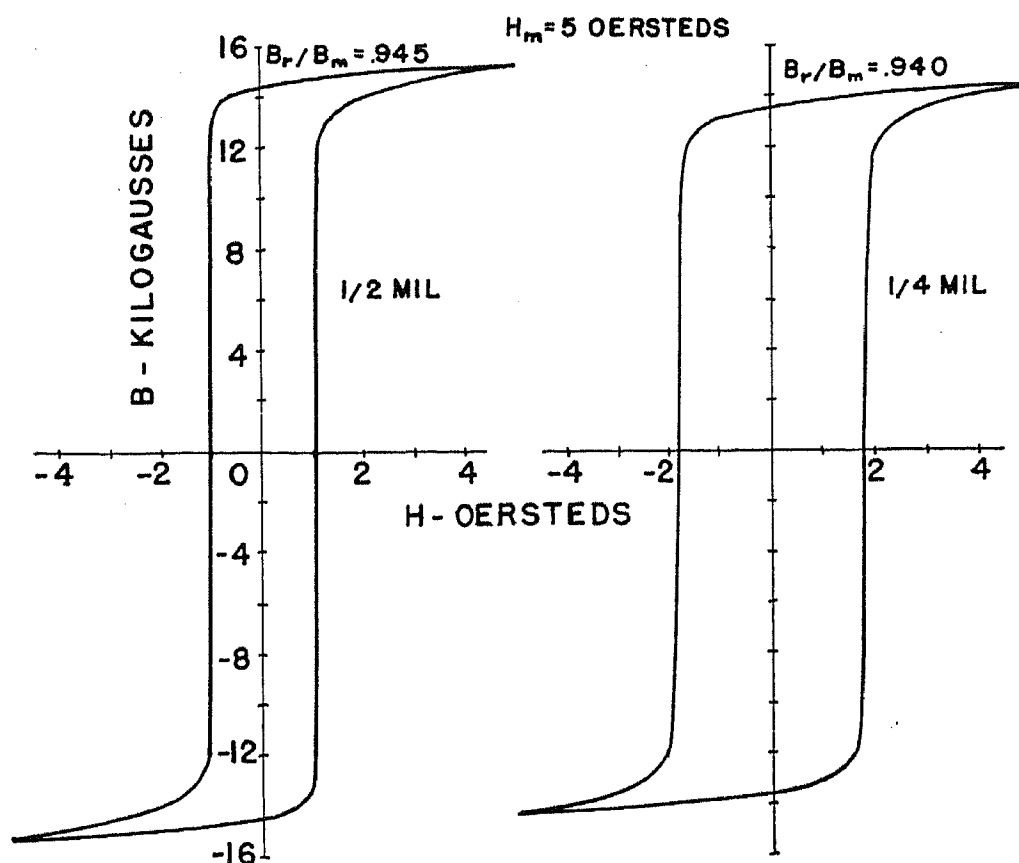


Figure 2 (above). D-c magnetization curves of 1/4-mil ultrathin metal tapes compared with other materials used at high frequencies

Figure 3 (right). D-c hysteresis loops of grain-oriented 3-per-cent silicon-iron



cores results in very low d-c permeability even for the best alloys. Where high permeability is desirable, very thin metal tapes are useful and these are the basis of the present paper.

The commercial use of magnetic alloys rolled to thin gauges grew very rapidly during World War II. For power transformers, laminated or tape-wound cores of strip 4- to 6-mils thick were employed, and for radar pulse transformers cores of 1- to 3-mil-thick materials were used. This large-scale use of thin materials was, to a large degree, the result of development of economical methods of rolling 1- to 6-mil strip, primarily silicon-iron, in quantities of thousands of pounds. Already during this period there were some applications involving rates of magnetization so high that materials even thinner than 1 mil (0.025 millimeter) were required. Small quantities of 4-79 Permalloy tapes 1/4-mil thick were rolled on a small mill at the Bell Telephone Laboratories for such applications.<sup>2</sup> The production of magnetic alloys in tapes as thin as 1/8 mil has now been undertaken by commercial organizations.

Since 1950, four different alloys have been rolled to gauges less than 1 mil and fabricated into toroidal cores. The descriptive term ultrathin has been applied to these materials. The alloys chosen for study are commonly used soft magnetic materials having high permeability. The group was restricted to those alloys having relatively high resistivity in order to keep eddy-current effects to a minimum. The values of magnetic saturation, Curie temperature, and volume resistivity, as shown in Table I, are characteristic of the alloys themselves and probably do not change with thickness in the range of 1 mil to 1/8 mil.

The alloys were rolled as single strands in a width of 2 inches employing work rolls of very small diameter. After rolling, the strip was slit into ribbons 1/8-inch or 1/4-inch wide for winding into

toroidal cores. Because of the fragile nature of small toroidal cores, often consisting of only a few wraps of tape, it was advantageous to wind the cores on ceramic bobbins as shown in the foreground

Table I. Basic Properties of Alloys Investigated

	3-Per-Cent Si, Balance Fe	48-Per-Cent Ni, Balance Fe	79-Per-Cent Ni, 4-Per-Cent Mo, Balance Fe	79-Per-Cent Ni, 5-Per-Cent Mo, Balance Fe
Magnetic Saturation, Gauss.....	20,000.....	16,000.....	8,700.....	8,000+
Curie Temperature, Degrees Centi- grade.....	700.....	440.....	420.....	400
Resistivity, Microhm-Centimeters.....	47.....	45.....	55.....	60
Density, Grams per Cubic Centi- meter.....	7.65.....	8.25.....	8.72.....	8.77

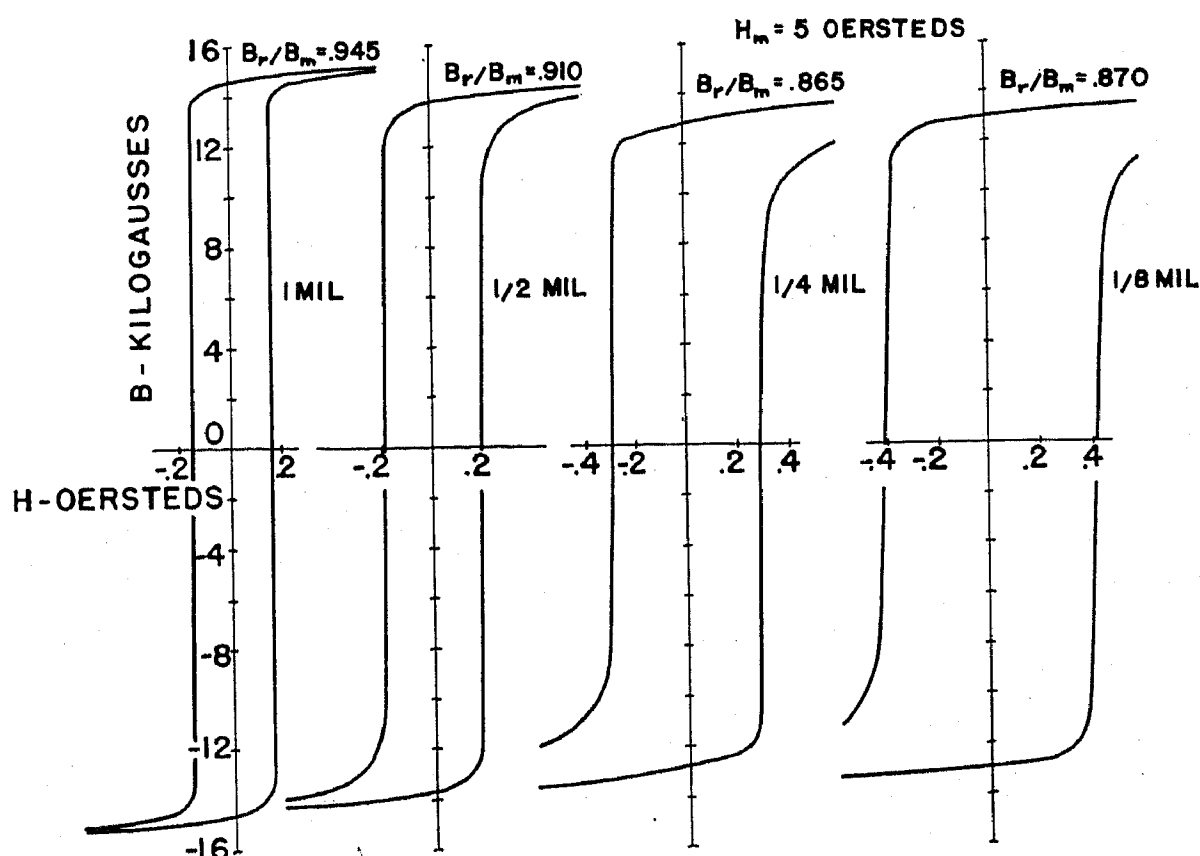


Figure 4. D-c hysteresis loops of grain-oriented 48-per-cent nickel-iron

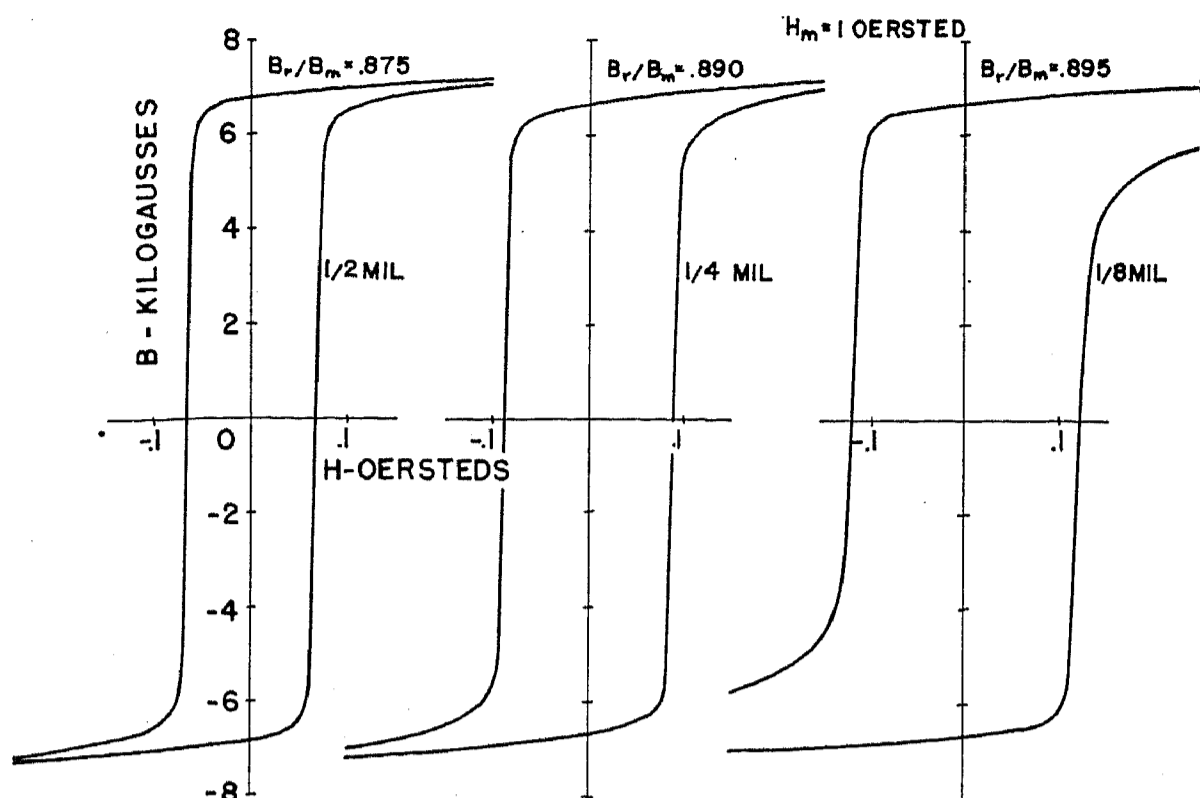


Figure 5. D-c hysteresis loops of 4-79 Permalloy showing pronounced rectangularity

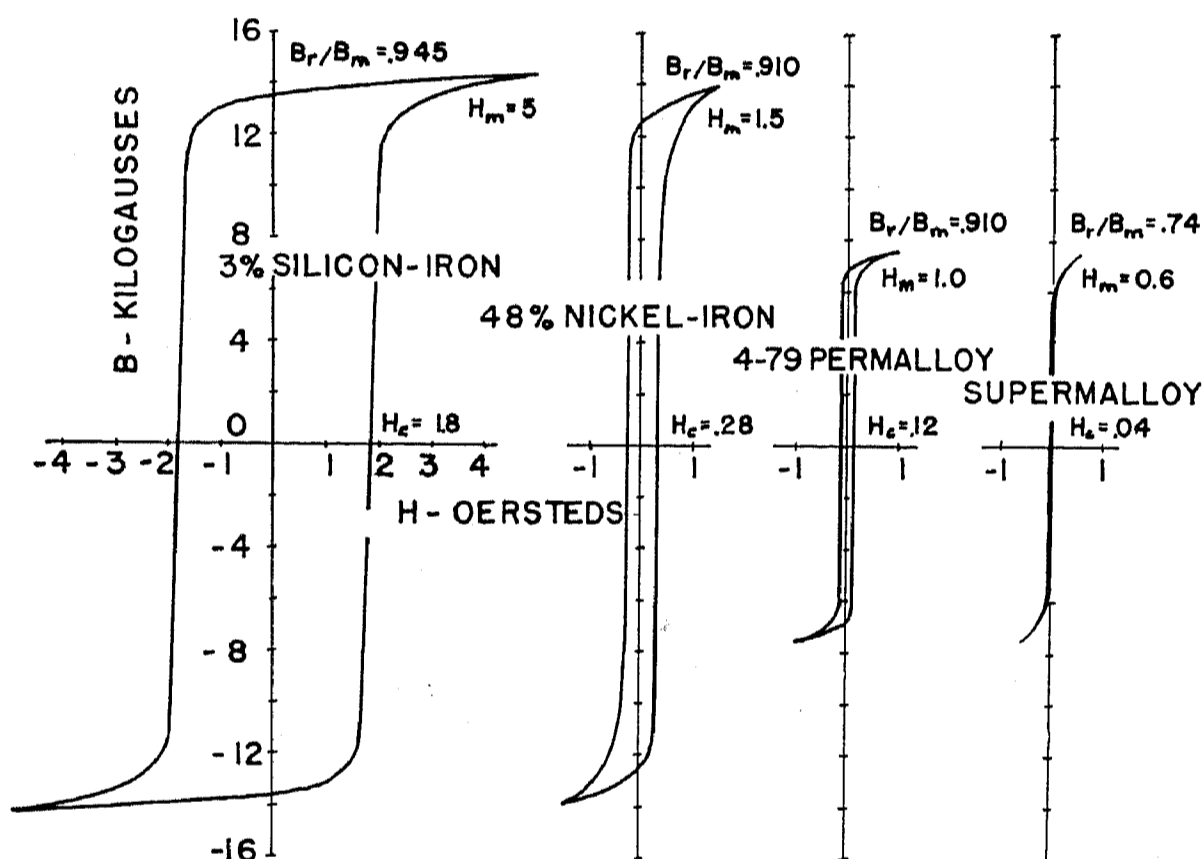


Figure 6. Comparison of hysteresis loops for ultrathin magnetic alloys 1/4-mil thick. Data plotted to same scale

of Figure 1. The bobbins furnished support for the tape during annealing and also provided a frame for the copper windings used in testing. The tape was insulated with a magnesia coating prior to annealing. Annealing was conducted in very dry, pure hydrogen at temperatures from 875 to 1,150 degrees centigrade. The particular magnetic properties obtainable depend not only on the composition of the alloy but on the processing steps of the metal supplier, the forming and annealing procedure used by the fabricator, and the stresses and tempera-

ture to which the core may be subjected after it is completed. These factors, individually or collectively as the case may be, all contribute to the resistivity, crystal and domain orientation, and stresses present in the lattice structure of the completed core and thus govern the characteristics of the core material when a magnetomotive force is applied. Naturally it would be most difficult to present a detailed picture of the magnetic properties of ultrathin magnetic alloys at this early stage of development. Our purpose is to present a general picture of the type of

magnetic characteristics we may expect under d-c magnetization as determined by the ballistic galvanometer. The measurements were made at room temperature with the flux parallel to the rolling direction of the strip.

The d-c permeability, particularly at high inductions, of ultrathin metal tapes is very much higher than for other materials used for high frequencies as shown in Figure 2. It may be noted that the d-c magnetization curves for the 1/4-mil tapes are very similar to those for the same alloys in the more familiar thicknesses of 1 to 14 mils. For comparison, the magnetization curve is shown for a specimen of 1-mil, grain-oriented 48-per-cent nickel-iron having a very rectangular hysteresis loop. Rectangular hysteresis loops are of considerable interest for applications such as magnetic amplifiers and memory cores and, as will be shown, ultrathin metal tapes may be made to have hysteresis loops which are quite rectangular.

For example, the loops shown in Figure 3 for 1/2-mil and 1/4-mil grain-oriented\* 3-per-cent silicon-iron show a high degree of rectangularity. Comparing the width of the two loops it may be noted that the coercive force is almost doubled by the reduction in thickness from 1/2 to 1/4 mil. At first thought this might be considered to be the result of an increased volume effect of surface contamination during final annealing. Of course, these very thin tapes are particularly sensitive to damage from faulty atmosphere during annealing. However, the increase in coercive force with reduced thickness has been observed even under very carefully controlled annealing conditions. The suggestion is made that as thickness is reduced there are changes in domain size and shape which cause an increase in coercive force. A contributing factor may be the type of crystal orientation present, particularly for material such as the 3-per-cent silicon-iron, which has a high coefficient of anisotropy.

As shown in Figure 4, the coercive force for grain-oriented† 48-per-cent nickel-iron also increases with reduced thickness but not as rapidly as for the silicon-iron. Compared to the 3-per-cent silicon-iron the coercive force for 48-per-cent nickel-iron is roughly 10 times lower for thicknesses below 1 mil.

4-79 Permalloy is generally regarded as a material having high permeability particularly at low inductions. Hence, the

\*Substantially that of a single crystal with a cube edge in the rolling direction and a face diagonal in the transverse direction.

†Substantially that of a single crystal with a cube edge both in the rolling and transverse directions.

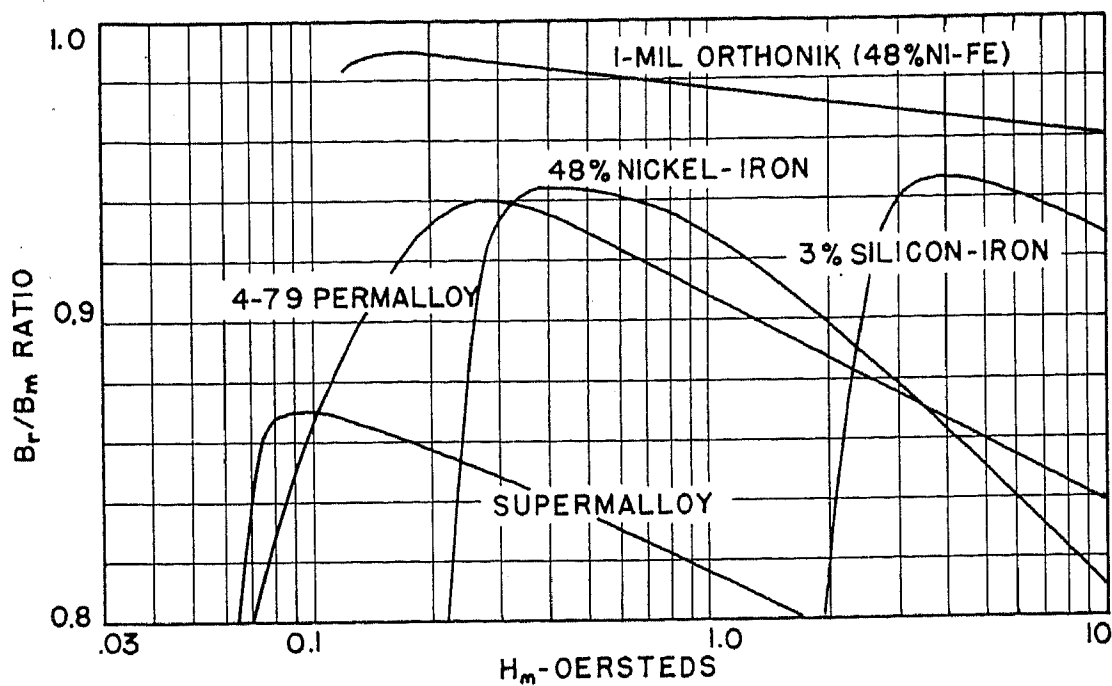


Figure 7. Variation of ratio of residual to maximum induction with magnetizing force for 1/4-mil cores with rectangular hysteresis loops. Data for 1-mil grain-oriented 48-per-cent nickel-iron included for comparison

high degree of rectangularity of the hysteresis loop which it was possible to obtain in ultrathin gauges of 4-79 Permalloy was hardly expected, see Figure 5.

The hysteresis loops for 1/4-mil experimental specimens of the various alloys including Supermalloy are compared with each other on the same scale in Figure 6 which illustrates how the materials compare in a qualitative way. The comparison is intended to be on a qualitative basis since it will be understood that the properties for each alloy can be varied considerably and the particular specimens, data for which are presented here, were chosen because their hysteresis loops were quite rectangular. Again referring to Figure 6, the various shapes of loops illustrate that the terms "square loop" or rectangular hysteresis loop are, at best, only

descriptive and that important differences in degree exist depending on the criteria used. Such criteria may be the slopes of the sides of the loops, or the slope of the top of the loop as saturation is approached.

One useful way of measuring rectangularity is to note how the ratio of residual to peak or maximum induction  $B_r/B_m$  varies with peak or maximum magnetizing force  $H_m$ . The curves for  $H_m$  versus  $B_r/B_m$  shown in Figure 7 represent the same cores used for Figure 6 and Figure 2. Lower ratios of  $B_r/B_m$  may be achieved by variation of processing and annealing treatments. Very low ratios may be obtained by the introduction of a small air gap into the magnetic path. Supermalloy is the only material among those studied here in 1/4-mil thickness which has not been found thus far to display a

maximum ratio of  $B_r/B_m$  greater than 0.9.

The space factor or packing factor present in the experimental cores used for the tests described above is as low as 50 per cent for 1/4-mil strip. For small cores containing only a few turns of strip, space factor is not a limitation. Where space and weight are limiting factors, the thickness of magnesia insulation would probably be a handicap. However, there is considerable promise that improved techniques may permit much higher space factors without impairing the value of the insulation.

It is too early to state in detail what the potential capabilities of these ultrathin materials are under pulse excitation or continuous high-frequency operation. The 48-per-cent nickel-iron and 4-79 Permalloy tapes show considerable promise for high-speed memory cores in computers. For magnetic amplifiers and radar pulse transformers, the low coercive force and high magnetic saturation of ultrathin alloy tapes are of interest, although it may not be desirable necessarily to have a rectangular d-c hysteresis loop. Ultra-thin 4-79 Permalloy appears to be particularly attractive since it has relatively high volume resistivity and can be made to have either a rectangular or rounded d-c hysteresis loop together with low coercive force.

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## No Discussion

# The Current Status of Dynamic Stability Theory

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**A**LTHOUGH there exists today no theory of dynamic stability of sufficient generality to satisfy the demands of engineering, there are, nevertheless, several tools which may be employed advantageously by the engineer in many

stability investigations. A short description of these tools illustrated by certain practical examples will be presented.

The concepts of absolute and relative stabilities of dynamical systems are matters of much concern in nearly all branches

of the physical sciences and in many branches of the social sciences. Although a great amount of research effort has been expended in this field, it can be said safely that only the simplest stability problems have been solved satisfactorily. This statement applies not only to nonlinear systems, but also to many systems which

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can be represented in their ranges of operation by entirely linear equations, as, for example, the linear differential equation with periodic coefficients.

The fundamental problem is the definition of stability. It is difficult to find one definition which will suit all purposes. Asymptotic stability, as formulated by Liapounoff,<sup>1</sup> is a satisfactory tool for the investigation of the action of a regulator or a feedback amplifier but is unsatisfactory in the treatment of conservative systems, unless all periodic behavior is termed unstable. Liapounoff's criterion of stability of so-called steady motion rectifies this difficulty for cases (such as linear conservative systems) in which the period of oscillation is independent of amplitude. However, such a criterion applied to the motion of celestial bodies would produce the embarrassing conclusion that the motion of each planet is one of unstable equilibrium. Thus it is necessary to introduce a third concept; namely orbital stability.<sup>2</sup>

The three concepts, asymptotic stability, orbital stability, and stability in the sense of Liapounoff (hereafter to be referred to as temporal stability) may be termed absolute stabilities. They do not provide, in themselves, a solution to all problems of definition. A servomechanism, although absolutely and asymptotically stable, may exhibit such a persistent oscillatory response as to render its usefulness no greater than if it had been unstable. Another concept must be introduced, therefore, and one is concerned with degree of damping or relative stability.

Although too little damping is objectionable in many systems, in a few cases zero damping or even a slight negative damping is permissible. Consider the case of the guided missile, the flight path of which is often characterized by oscillations occasionally increasing in amplitude. Inasmuch as the total flight time of the missile may include no more than three or four oscillations, the magnitude of the

negative damping is the proper criterion of merit of the control system and the concept of absolute stability is not important.

## Methods

Three types of motion will be discussed in the order of increasing complexity:

1. Stationary points, or points of equilibrium, such as operating points of regulatory devices.
2. Periodic motions such as occur in the operation of the electric oscillator.
3. Any bounded or unbounded, periodic or nonperiodic motion.

Obviously item 3 includes 2, and item 2 includes 1. However, items 1 and 2 are more commonly encountered than is item 3, and there is some advantage in a separate treatment.

The particular motion, the stability of which is under investigation, whether it be an equilibrium point or a periodic or nonperiodic wave form, is generally denoted as the undisturbed motion, whereas any other motion of the dynamical system is called a disturbed motion. Broadly speaking, a system is said to be asymptotically stable if all disturbed motions sooner or later degenerate into the undisturbed motion itself. Application of the phrase "all disturbed motions" must be made with reservation however. Consider the case of the periodic motion of a wrist watch. Any light shock will change its period momentarily, but eventually the motion reverts to its former value, and consequently the timepiece is labeled stable. However, if dropped from a second story window, the watch quickly reaches a condition of instability.

The situation is exemplified by the electric oscillator which will not oscillate unless first dropped on the floor, or perhaps by the more practical case of the amplifier which "sings" when overloaded and continues in this state after removal of the excitation. These two examples refer to a

type of behavior called hard excitation, in which the system becomes unstable only when subjected to disturbances of sufficiently large value. The general problem of dynamic stability under hard excitation is very difficult. Any one problem usually can be solved only by obtaining an approximate solution for the motion of the system subject to all possible boundary conditions.

Liapounoff presents two methods for the investigation of stability under soft excitation, that is, stability of dynamical systems subject only to infinitesimal disturbances. These are called the first and second methods. The former involves the investigation of the behavior of all solutions which originate in an infinitesimal region surrounding the undisturbed motion. The second method involves the investigation of the existence of a function which, in connection with the given dynamical equations, possesses a certain prescribed behavior. The difficulty in finding such restricted functions has resulted in very little use of the second method, and consequently stability investigations are effected by means of the first method.

## Definition

Consider a dynamical system described by the set of differential equations

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \dots, x_n), \quad i = 1, 2, \dots, n, \quad (1)$$

in which the functions  $f_i$  are, in general, nonlinear. Let  $x_{i0}(t)$  denote a particular undisturbed motion, and let  $x_i(t)$  denote any disturbed motion. The undisturbed motion is temporally stable if, and only if, for a given positive number  $\epsilon$ , no matter how small, it is possible to find another number  $\delta$  such that for all disturbed motions satisfying the initial inequality

$$|x_{i0}(0) - x_i(0)| < \delta \quad (2)$$

the subsequent motion satisfies the inequality

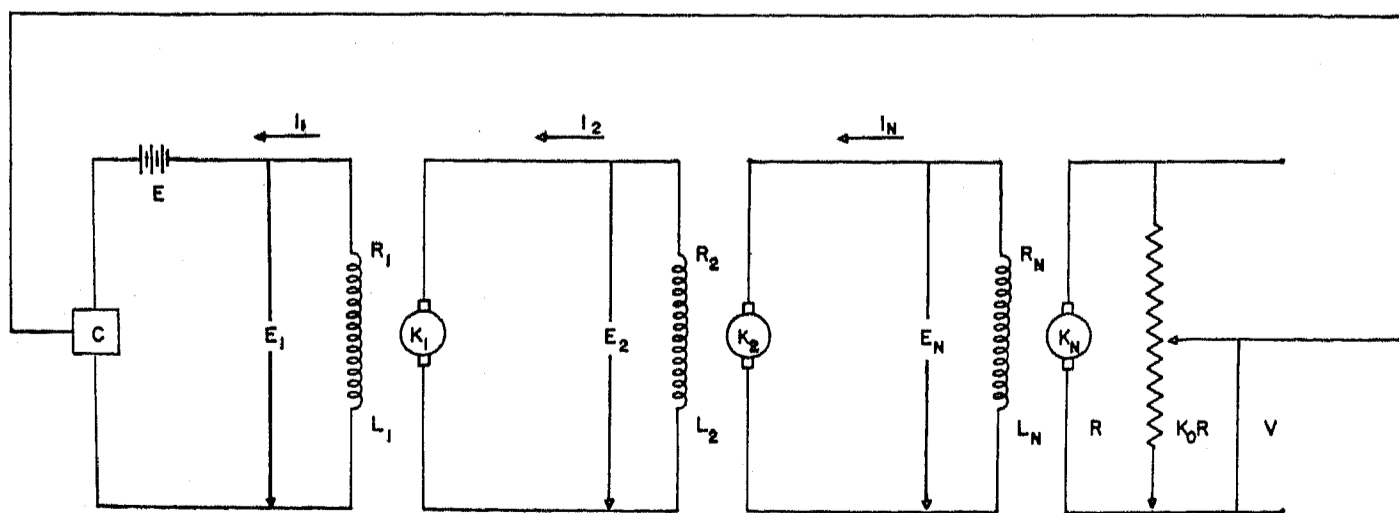


Figure 1. Voltage regulated n-stage d-c generator

$$|x_{i0}(t) - x_i(t)| < \epsilon \quad (3)$$

If, in addition

$$\lim_{t \rightarrow \infty} |x_{i0}(t) - x_i(t)| = 0 \quad (4)$$

the undisturbed motion is said to be asymptotically stable. If the undisturbed motion  $x_{i0}(t)$  is periodic, equation 3 requires the approach of the disturbed motion to a motion of exactly the same period as the period of the undisturbed motion. This last requirement may be relaxed by means of the concept of orbital stability. However, this particular type of stability will not be considered in this paper.

## The First Approximation

The tool which so far has been found most useful (and which falls in the category of the first method) in the investigation of dynamic stability, is the so-called method of first approximation. The procedure is contained in the following six steps.

### REPRESENTATION OF THE PHYSICAL SYSTEM BY MEANS OF A SET OF DYNAMICAL EQUATIONS

This first step is often the most difficult. There is no general method of approach, although the resultant dynamical representation is invariably in the form of differential, difference, or integral equations, or, in some cases, a mixed system involving two or three types.

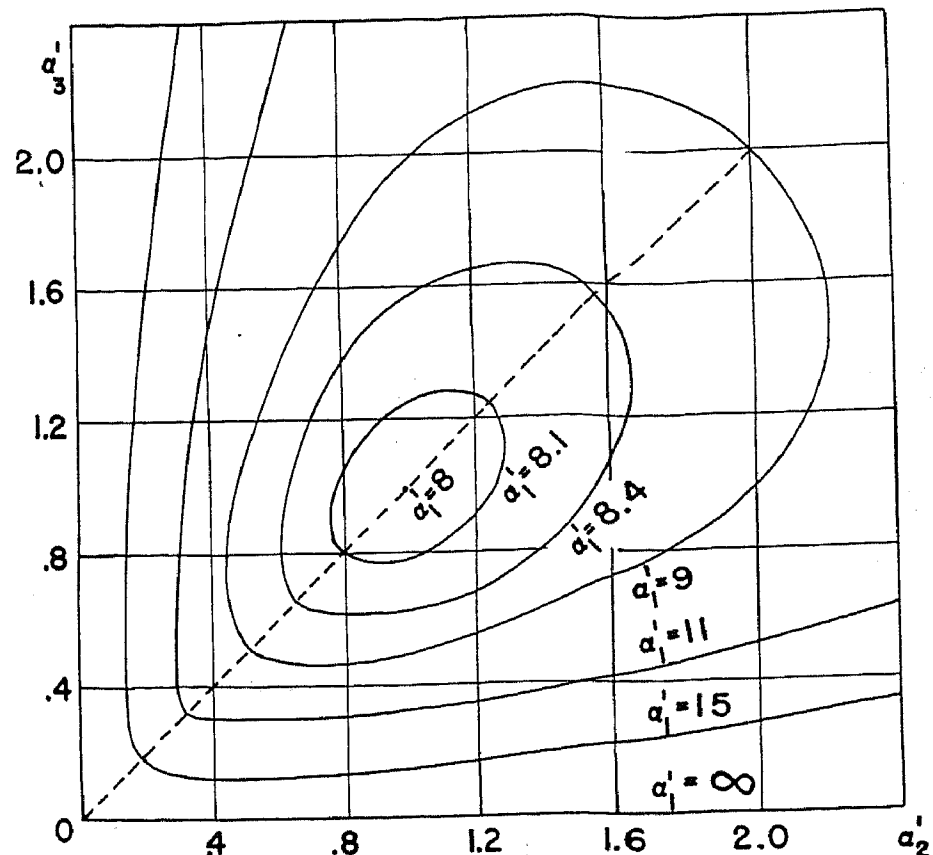
### SEARCH FOR THE UNDISTURBED MOTIONS

If the undisturbed motions are stationary points, they are easily found by solving the static equations obtained from the dynamic equations by deleting all derivatives and differences. However, if the undisturbed motions are periodically varying solutions of the dynamical representation, their determination is often most difficult. An approximate means of solution will be provided in the section on the method of equivalent linearization. When the undisturbed motion is nonperiodic and is prescribed by means of certain boundary or other characterizing conditions, there is no general method of approach for nonlinear systems.

### PERTURBATIONS

Once the undisturbed motion has been found, all disturbed motions in the neighborhood of the undisturbed motion may be found by means of perturbations. The usual method involves the expansion of the original set of dynamical equations in

Figure 2. Generalized stability boundary



Taylor series about the undisturbed motion itself. Stability of the set of linear perturbation equations resulting from neglect of the higher order Taylor series terms generally, but not always, guarantees stability of the original dynamical equations. The cases in which the rule breaks down are usually those in which certain linear perturbation terms disappear. Inasmuch as one is usually concerned with the effect on absolute and relative stabilities of system parameter variations, and since the rule breakdown generally occurs only when the system parameters are such that the system itself is on the very edge of instability, such an occurrence requires no further attention. However, there are cases in which the breakdown of the rule occurs for all parameter values, the method of first approximation fails, and one is forced to consider higher order Taylor series terms.

### CHARACTERISTIC EQUATION

If the original dynamical equations are autonomous, that is, if the dependence on time is not explicit, and if the undisturbed motion is a stationary point, the equations of first approximation are linear and possess constant coefficients. Stability is consequently determined by the roots of the characteristic equation of the first approximation. If the original system is described entirely by differential equations or entirely by difference equations, the characteristic equation is either an algebraic equation or can be made so by a logarithmic transformation. However, if the system is represented by a mixed system of differential-difference equations, the characteristic equation is transcen-

dental and presents certain difficulties of solution.

### STABILITY CRITERIA

The occurrence of one or more roots of the characteristic equation with positive real part is the necessary and sufficient condition for instability of the original system. Whether or not this situation occurs can be determined by means of either the numerical method of Nyquist<sup>3</sup> or the analytical method of Routh<sup>4</sup> and Hurwitz.<sup>5</sup> The choice of method depends on the complexity of the characteristic equation and on whether one is concerned with parameter variations.<sup>6</sup> Both the Routh method and the Nyquist method were originally derived for the treatment of absolute stability. However each may be modified to yield certain information concerning the actual location of the roots.<sup>6-9</sup>

### STABILITY BOUNDARIES

The final step in any stability analysis is the presentation of the absolute and relative stability boundaries as functions of basic system parameters. A few examples will now be given.

### Voltage Regulation<sup>10</sup>

Let us follow through the six steps for the  $n$ -stage d-c generator, shown in Figure 1, which employs a 3-terminal nonlinear regulator  $C$ , such as the step resistor, the resistance value of which varies in steps with the control voltage  $V$ .

1. Denoting the voltage function of the regulator  $C$  by  $f(I_1, V)$ , the dynamical equations are

$$E = E_1 + f\left(\frac{E_2}{K_1 R_1}, V\right)$$

$$dE_2/dt = (R_1/L_1)(K_1 E_1 - E_2)$$

$$dE_3/dt = (R_2/L_2)(K_2 E_2 - E_3)$$

$$\dots$$

$$dE_n/dt = (R_{n-1}/L_{n-1})(K_{n-1} E_{n-1} - E_n)$$

$$dV/dt = (R_n/L_n)(K_n E_n - V)$$

2. The undisturbed motion is a stationary point, the operating point of the system. It is obtained by deleting the derivative terms in equation 5 and solving the resulting equations. Denoting the equilibrium voltages by subscript zero, it may readily be shown that

$$E_{20} = K_1 E_{10}$$

$$E_{30} = K_1 K_2 E_{10}$$

$$\dots$$

$$E_{n0} = K_1 K_2 \dots K_{n-1} E_{10}$$

$$V_0 = K_0 K_1 K_2 \dots K_n E_{10}$$

in which  $E_{10}$  is given by the nonlinear equation

$$E = E_{10} + f(E_{10}/R_1, K_0 K_1 \dots K_n E_{10}) \quad (7)$$

3. Setting  $E_i = E_{i0} + e_i$  and  $V = V_0 + v$ , one finds the set of linear perturbation equations

$$e_1 + (f_{I_1}/K_1 R_1)e_2 + f_V v = 0$$

$$de_2/dt + (R_1/L_1)e_2 - K_1(R_1/L_1)e_1 = 0$$

$$\dots$$

$$de_n/dt + (R_{n-1}/L_{n-1})e_n - K_{n-1} \times$$

$$(R_{n-1}/L_{n-1})e_{n-1} = 0$$

$$dv/dt + (R_n/L_n)v - K_n K_{n-1} (R_n/L_n)e_n = 0$$

in which  $f_{I_1}$  and  $f_V$  denote the partial derivatives of  $f$  with respect to  $I_1$  and  $V$ , evaluated at the equilibrium point.

4. The characteristic equation 8 is obtained by equating to zero the determinant of the coefficients of the unknown voltages  $e_i$  and  $v$ , after replacing the operation of differentiation by the symbol  $p$ . It is

$$\left(p + \frac{R_n}{L_n}\right) \left(p + \frac{R_{n-1}}{L_{n-1}}\right) \dots \left(p + \frac{R_2}{L_2}\right) \times$$

$$\left[p + \frac{R_1}{L_1} \left(1 + \frac{f_{I_1}}{R_1}\right)\right] +$$

$$K \frac{R_1 R_2 \dots R_n}{L_1 L_2 \dots L_n} f_V = 0 \quad (9)$$

in which  $K = K_0 K_1 \dots K_n$

5. The Hurwitz determinant is useful in this case. For a 3-stage generator, equation 9 is the cubic

$$a_0 p^3 + a_1 p^2 + a_2 p + a_3 = 0 \quad (10)$$

in which

$$a_0 = 1$$

$$a_1 = (R_3/L_3) + (R_2/L_2) + (R_1/L_1) \left(1 + \frac{f_{I_1}}{R_1}\right)$$

$$a_2 = \frac{R_3 R_2}{L_3 L_2} + \left(\frac{R_3}{L_3} + \frac{R_2}{L_2}\right) \frac{R_1}{L_1} \left(1 + \frac{f_{I_1}}{R_1}\right)$$

$$a_3 = \frac{R_3 R_2 R_1}{L_3 L_2 L_1} \left(1 + \frac{f_{I_1}}{R_1} + K f_V\right) \quad (11)$$

The stability boundary is obtained by equating to zero the penultimate Hurwitz determinant

$$\begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}$$

Using equation 11, one finds an expression for the stability boundary

$$(1 + \alpha_2' + \alpha_3') \left(1 + \frac{1}{\alpha_2'} + \frac{1}{\alpha_3'}\right) = 1 + \alpha_1' \quad (12)$$

shown in Figure 2, in which

$$\alpha_1' = K \frac{f_V}{1 + \frac{f_{I_1}}{R_1}}, \quad \alpha_2' = \frac{R_2}{L_2} \frac{L_1}{R_1 + f_{I_1}},$$

$$\alpha_3' = \frac{R_3}{L_3} \frac{L_1}{R_1 + f_{I_1}} \quad (13)$$

The region of stability lies below the surface considered as a contour map.

6. Except for the case of a linear control element  $C$ , the partial derivatives  $f_{I_1}$  and  $f_V$  are functions of the location of the operating point, which itself is determined by the circuit parameter values. Therefore equation 12 and Figure 2 do not express, in general, the stability boundary explicitly in terms of basic system parameters, and, in fact, the problem cannot be completely solved until the form of the control function  $f(I_1, V)$  is known. A common control element for high-voltage generators is the step resistor. The essential features of its stability in connection with Figure 1 may be observed by assuming that the resistance varies linearly and continuously with control voltage when the latter exceeds a critical value,  $V_c$ . Thus

$$R_c = 0 \quad \text{if } V < V_c$$

$$= K_c \frac{R_1}{V_c} (V - V_c) \quad \text{if } V > V_c$$

so that

$$f(I_1, V) = 0 \quad \text{if } V < V_c$$

$$= K_c \frac{R_1}{V_c} (V - V_c) I_1 \quad \text{if } V > V_c$$

and the stability boundary takes the form of Figure 3, or

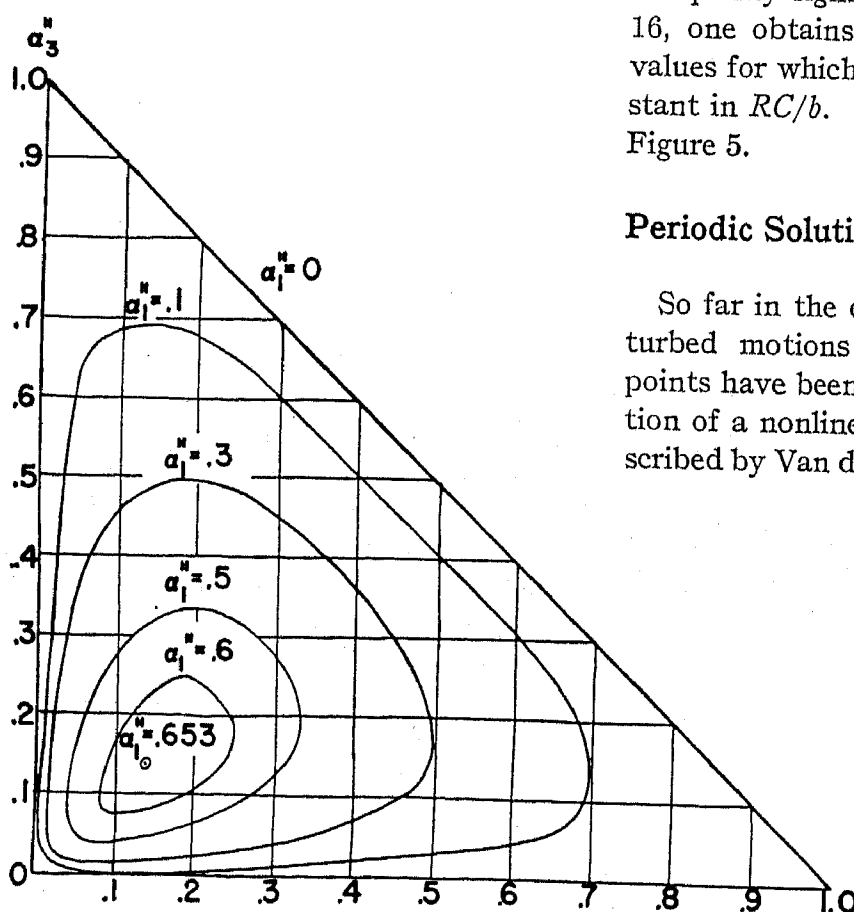


Figure 3. Stability boundary for a controlled resistance as control element

$$\left[\frac{1}{2}(\sqrt{1 + \alpha_1''} - 1) + \alpha_2'' + \alpha_3''\right] \times$$

$$\left[1 + \frac{1}{2}(\sqrt{1 + \alpha_1''} - 1) \times$$

$$\left(\frac{1}{\alpha_2''} + \frac{1}{\alpha_3''}\right)\right] = \sqrt{1 + \alpha_1''} \quad (14)$$

in which

$$\alpha_1'' = 4 \frac{K_c}{(K_c - 1)^2} K_0 K_1 \dots K_n \frac{E}{V_c}$$

$$\alpha_2'' = \frac{1}{K_c - 1} \frac{R_2 L_1}{L_2 R_1} \quad \alpha_3'' = \frac{1}{K_c - 1} \frac{R_3 L_1}{L_3 R_1}$$

$$(15)$$

In Figure 3, the region of instability lies inside the mound enclosed by the coordinate plane  $\alpha_1'' = 0$  and the contour surface.

## Shunt Feedback Amplifier<sup>6</sup>

An excellent example of the concept of relative stability is afforded by the treatment of the circuit of Figure 4. For the case in which the admittance  $Y$  is composed of a parallel combination of resistance and capacitance  $R$  and  $C$ , one system time constant is always equal to  $RC$ . Denoting by  $RC/b$ , where  $-\infty < b < 1$ , the value of the maximum time constant of the system, the region of relative stability is given by the inequality

$$\alpha_3 < \frac{1}{\alpha_1 - \alpha_2} \quad \text{for } \alpha_2 < \frac{2}{3} \alpha_1$$

$$< \frac{2(2 + \alpha_1 \alpha_2)^2}{2\alpha_1 + \alpha_2} \quad \text{for } \alpha_2 > \frac{2}{3} \alpha_1 \quad (16)$$

in which

$$\alpha_1 = K_2, \quad \alpha_2 = \frac{K_1 R S_3}{1 - b}, \quad \alpha_3 = \frac{R^2 S_1 S_2}{(1 - b)^2}$$

If equality signs are inserted in equation 16, one obtains the locus of parameter values for which the maximum time constant in  $RC/b$ . The results are shown in Figure 5.

## Periodic Solutions

So far in the examples only the undisturbed motions which were stationary points have been considered. The operation of a nonlinear oscillator is often described by Van der Pol's equation

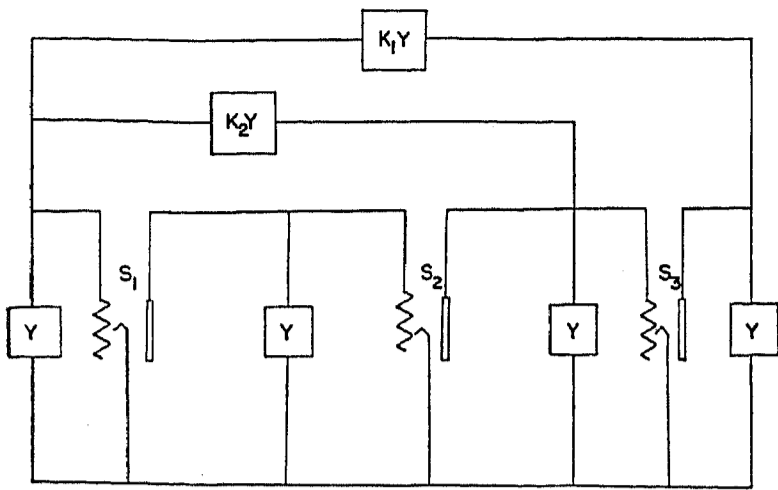
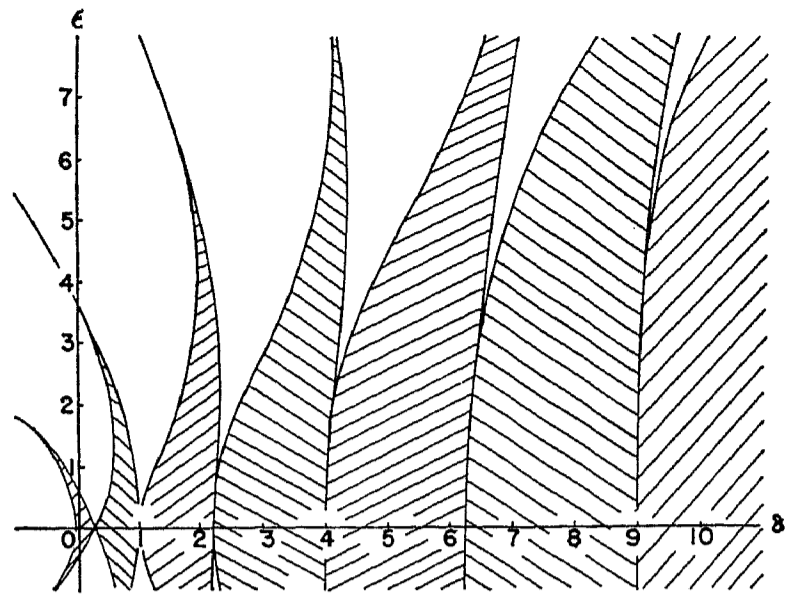


Figure 4 (left).  
Illustrative circuit—shunt feedback amplifier

Figure 6 (left).  
Stable and unstable regions of the Mathieu equation



$$\frac{d^2x}{dt^2} + x + \mu(x^2 - 1) \frac{dx}{dt} = 0 \quad (17)$$

For  $\mu$  small, one undisturbed motion is given approximately by

$$x_0(t) = 2 \sin t \quad (18)$$

The equations of first approximation are obtained by the substitution of

$$x(t) = x_0(t) + \delta(t)$$

together with equation 18, into equation 17 and omitting all but linear terms in  $\delta$ . The result is the linear differential equation with periodic coefficients

$$\frac{d^2\delta}{dt^2} + \mu(4 \sin^2 t - 1) \frac{d\delta}{dt} + (1 + 8 \sin t \cos t) \delta = 0 \quad (19)$$

Here the methods of Nyquist and Routh are of no use, and the general method, as outlined by Stoker,<sup>11</sup> for example, stops far short of the actual solution for stability boundaries. In fact, the boundaries can be obtained only after one has evaluated a certain infinite order determinant called a Hill determinant.

For the very simple case of the Mathieu equation

$$\frac{d^2x}{dt^2} + (\delta + \epsilon \cos t)x = 0 \quad (20)$$

which represents a variety of physical systems,<sup>12</sup> the stability boundary has been completely determined and is shown in Figure 6. Even in this simple case the boundary is a very complicated function of the parameters  $\delta$  and  $\epsilon$ .

### Equivalent Linearization

A very powerful tool for the investigation of stability of periodic motion is based on the so-called method of equivalent linearization as formulated by Kryloff and Bogoliuboff,<sup>13</sup> and apparently first linked with the Nyquist criterion by Kochenburger<sup>14</sup> in his treatment of the relay servomechanism. The method involves the systematic replacement of all nonlinear elements in the system by quasi-linear elements which have the property of preservation of waveform but nonlinear magnification. For example, in the Van der Pol equation 17, the non-

linear term

$$(x^2 - 1) \frac{dx}{dt} \quad (21)$$

is replaced by a linear term which is the result of considering only the fundamental term in the response of equation 21 to the excitation  $x = a \sin \omega t$ . But the fundamental term is  $(a^2/4 - 1)a\omega \cos \omega t$ , so that the equivalent linear term is  $(a^2/4 - 1)(dx/dt)$ , and equation 17 is replaced by

$$\frac{d^2x}{dt^2} + x + \mu \left( \frac{a^2}{4} - 1 \right) \frac{dx}{dt} = 0 \quad (22)$$

Thus oscillations which have amplitude greater than  $a = 2$  are obviously damped, while oscillations of magnitude less than  $a = 2$  obviously are characterized by negative damping. Therefore, the solution  $x = 2 \sin t$  is stable.

Inasmuch as the disturbed motion (and for that fact even the undisturbed motion) is not exactly sinusoidal, there is some doubt as to what is meant by the amplitude of motion. In fact, the whole procedure rests on rather flimsy mathematical ground. However, in systems which are essentially low-pass filters, such as servomechanisms, wherein harmonic distortion is low, the procedure has been very successful, and this is its great redeeming feature.

A remarkable example of the use of equivalent linearization occurs in the field of economics. A model of the business cycle has been hypothesized by Goodwin<sup>15</sup> in the form of the autonomous nonlinear mixed differential-difference equation

$$\gamma y(t + \theta) + (1 - \alpha)y(t) = \phi[y(t)] \quad (23)$$

in which  $y(t)$  denotes present national income, and the induced investment function  $\phi(x)$  is the unsymmetrical saturation curve of Figure 7. If  $c$  denotes the amplitude of the quasi-linear oscillation of  $y(t)$ , the characteristic equation of the

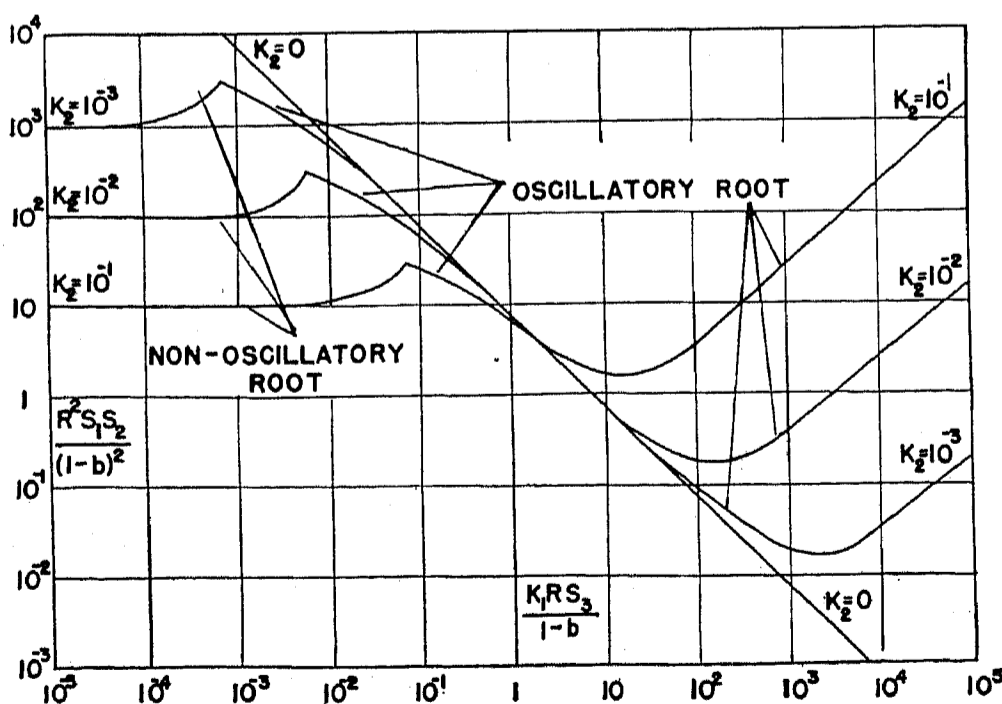


Figure 5. Locus of points for which the maximum time constant of Figure 4 is  $RC/b$

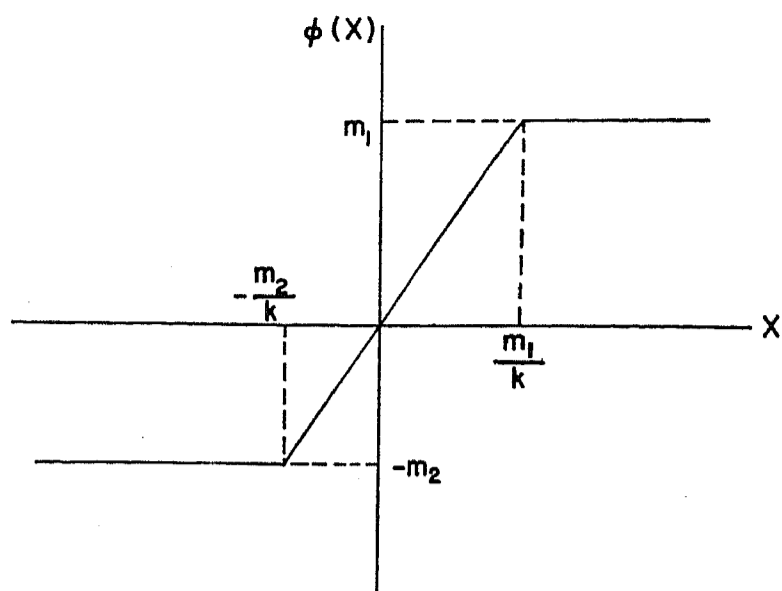
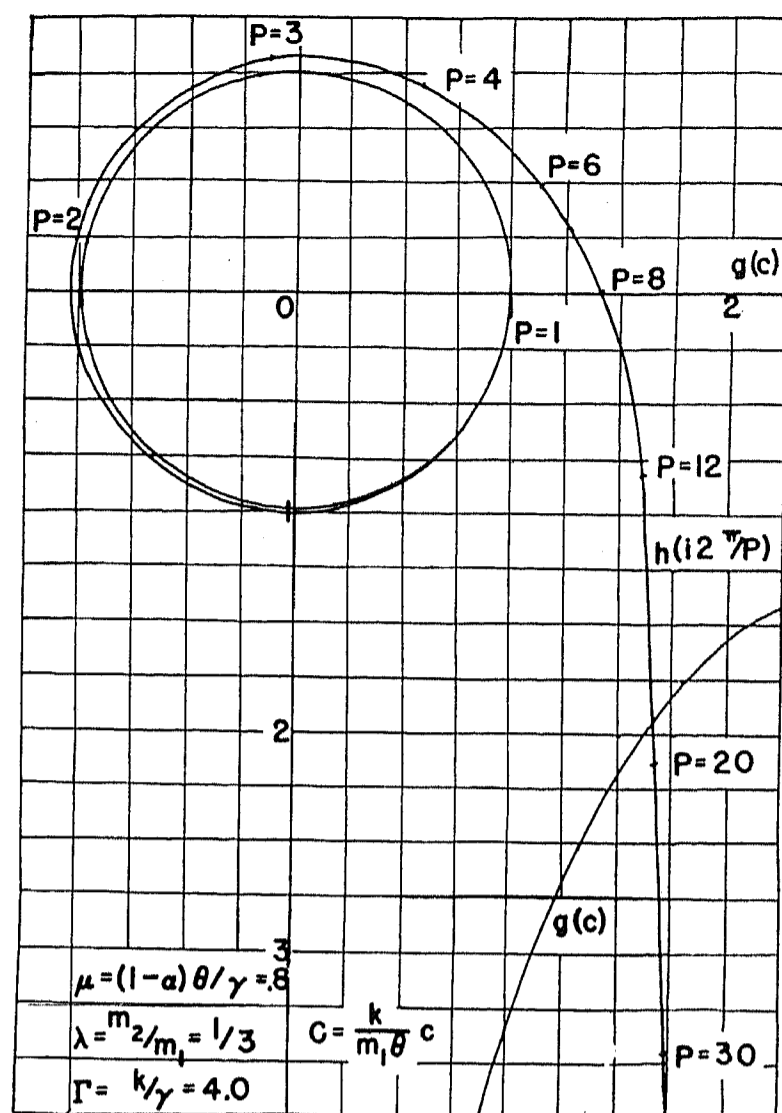


Figure 7 (left).  
Induced investment function

Figure 8 (right).  
Nyquist plot of  
quasi-linear representation of  
Goodwin's model of the business cycle



corresponding linearized equation is found to be of the form

$$h(z) = g(c) \quad (24)$$

The Nyquist plot of each side of equation 24 is shown in Figure 8. Here  $c$  is plotted as a function of  $g(c)$  so that the intersection of the spiral  $h(i\omega)$  ( $\omega = 2\pi/p$ , where  $p$  is the period of the assumed oscillation) with the  $g(c)$  axis determines an infinite number of admissible values of period  $p$ . That is, it is possible that the system oscillates in any one of an infinite number of modes. The admissible values of  $p$  are approximately 8 years, 1 year, 1/2 year, 1/3 year, and so forth.

The question of stability is decided by the following reasoning. For values of  $c$  for which the spiral encloses the point  $g(c)$ , the motion is stable,  $c$  will decrease, and  $g(c)$  will increase. For values of  $c$  for which the spiral does not enclose the point  $g(c)$ , the motion is unstable,  $c$  will increase, and  $g(c)$  will decrease. Obviously each of the infinite number of modes is stable. It is also apparent that a relatively small disturbance of a high order mode is sufficient to cause oscillation in a different mode, a result which is not available in the analysis of infinitesimal disturbances. That these modes do exist and are stable, has been demonstrated experimentally by means of analogue equipment.

## Conclusions

Of the three types of stability, orbital, temporal, and asymptotic, the last has received the most attention. Almost all the successful stability analyses have employed the first of Liapounoff's two methods, and have been concerned mainly with systems subject only to infinitesimal disturbances.

Undisturbed motions which are stationary points almost always are treated successfully by a sequence of first order perturbation and a stability criterion of Nyquist or Routh type. Relative stability as well as absolute stability may be treated by this method.

Undisturbed motions which are periodic are currently treated by two methods. The perturbation method invariably produces a linear equation with periodic coefficients, the stability of which may be investigated by Hill's method. However the general method stops far short of a solution, and only the simplest cases have been solved. Equivalent linearization has been currently successful in the analysis of low-pass systems. However, it is a very approximate method, refinements are involved, it fails completely on high-pass systems, and mathematical justification is lacking. The general problem of the stability of periodic motion remains unsolved.

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## No Discussion

# Types of Magnetic Amplifiers— Survey

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**Synopsis:** Many forms of magnetic amplifiers have been described in the past 50 years. The patents and writings relating to these devices have been examined to determine what types of magnetic amplifiers exist and what properties these devices have in common. The objects of this paper are: (1) to describe the various types of magnetic amplifiers, and (2) to present a method by which the basic principles relating to these devices may be codified according to an orderly system.

## Magnetic Amplifier Chart

**A** chart which is basic to a scheme for classifying the various types of magnetic amplifiers is given below. Any known magnetic amplifier may be represented by a combination of the key letters and numbers listed in the chart. Each of the letters and numbers represents information relating to the magnetic amplifier class, circuit type, and special means for affecting the characteristics of the particular amplifier to be described.

In defining classes A and B of the chart an arbitrary distinction has been made between inductive-impedance devices (A) in which the a-c power is applied to the core via the load or reactance winding, and (B) in which the a-c power is applied to the core via a separate primary winding. According to this distinction, therefore, key A refers to saturable reactors controlled by signal magnetomotive force and key B refers to saturable transformers controlled by signal magnetomotive force.

Two classes of signal-magnetomotive-force-controlled saturable transformers appear. The first, key B', relates to a class of saturable transformers in which control is effected by varying the incremental permeability or by otherwise distorting the flux-time wave form in the core. The second class, key B'', relates to the class of transformers in which control is effected by varying the reluctance of a magnetic path which shunts the transformer leg linking the secondary winding

and thus controlling the mutual coupling between the primary and secondary windings.

In defining the specific circuit types indicated on the chart, push-pull (symmetrical) circuits, key 1, are to be distinguished from single-sided (unsymmetrical) circuits, key 2.

It is to be understood that "direct current" is a generic term which represents the signal, bias, or control effects that actually may be transitory or otherwise variable with respect to time. A general characteristic of magnetic amplifiers is that the power-supply frequency is considerably higher than any component of input-signal frequency.

To simplify the classification, only single-stage magnetic amplifiers are considered. Cascaded arrangements of stages are not included, as such, in this classification, but considerations are stated which must be taken into account in the design of single magnetic-amplifier stages which are to be used in cascade with other stages.

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## Classification Chart for Magnetic Amplifiers

### General Classes

- A. Saturable reactors controlled by signal magnetomotive force
- B. Saturable transformers controlled by signal magnetomotive force
  - B'. Control by varying incremental permeability or by distorting wave form
  - B''. Control of mutual inductance by varying the reluctance of a saturable shunt
- C. Magnetic modulators with negative resistance characteristics

General classes A and B contain specific circuit types such as polarized and nonpolarized magnetic amplifiers comprising:

- 1. Single-sided circuits
- 2. Push-pull circuits
- 3. Wheatstone-bridge-type circuits
- 4. Special magnetic amplifier circuits (such as cross-connected pairs of magnetic amplifiers, magnetic amplifiers in combination with transformers, and so forth)

Amplifier sensitivity, stability, linearity, and speed of response may be affected by incorporating any one or more of the following means:

- a. External feedback (amplitude-related and/or phase-related external feedback means including impedance coupled means terminating in a single-control winding or special feedback winding)
- b. Internal feedback (by rectifiers seriesed with load windings)
- c. Unidirectional bias magnetomotive force of fixed magnitude
- d. A-c bias magnetomotive force at fundamental or harmonic of power supply (carrier) frequency
- e. Off-resonant and ferresonant means
- f. Frequency-discriminating means
- g. Additional co-operative wave-form distorting means
- h. Ultra-carrier frequency magnetomotive force means for reduction of hysteresis effects

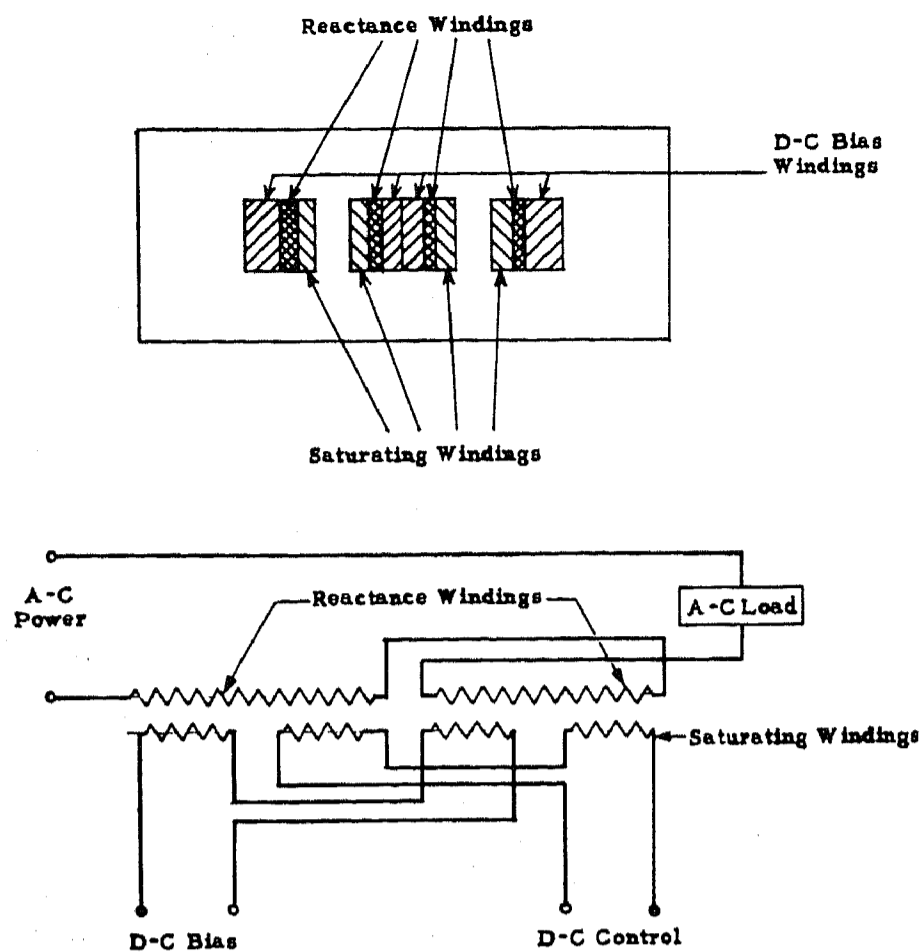


Figure 1. Single-sided d-c saturable reactor without feedback

### Examples of Types of Magnetic Amplifiers

Figures 1 through 20 show typical magnetic amplifiers which may be classified according to the chart. At least one example of a circuit relating to each key is shown in the figures. The examples have been chosen not on the basis of dates of invention, but because they illustrate particularly well the circuit types described.

Figure 1 (class A, 1, c) is a schematic of the first known d-c controlled saturable reactor for which gain was claimed. E. F. W. Alexanderson<sup>1</sup> described this device in United States Patent 1,206,643 (1916), and as a basic device it has been used widely. Alexanderson's device comprises two variable-reactance windings as load-current control means, the reactance

windings being placed in series between the a-c load and the a-c power supply. The total impedance of the circuit is controlled according to the reactance of the reactance windings. Control is accomplished in a well-known manner based on the change of incremental permeability of the core according to the amount of d-c saturating magnetomotive force applied to the core. The saturating windings are connected so that opposing a-c electromotive forces are induced in them by the reactance windings; the result is that zero alternating voltage of fundamental power-supply frequency appears across the terminals of the d-c control circuit. Alexanderson provided separate d-c bias windings for the purpose of introducing a d-c bias magnetomotive force to locate an optimum operating point on the magnetic characteristics.

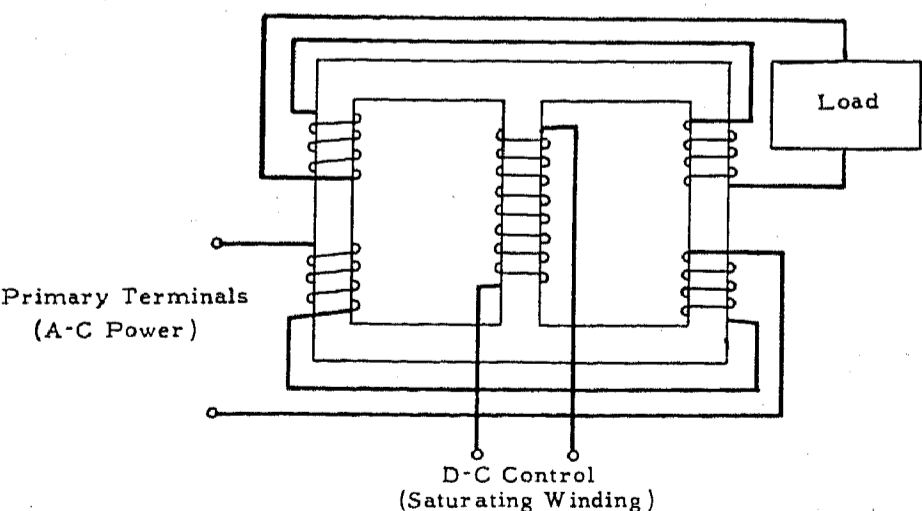


Figure 2. Variable-permeability type d-c saturable transformer

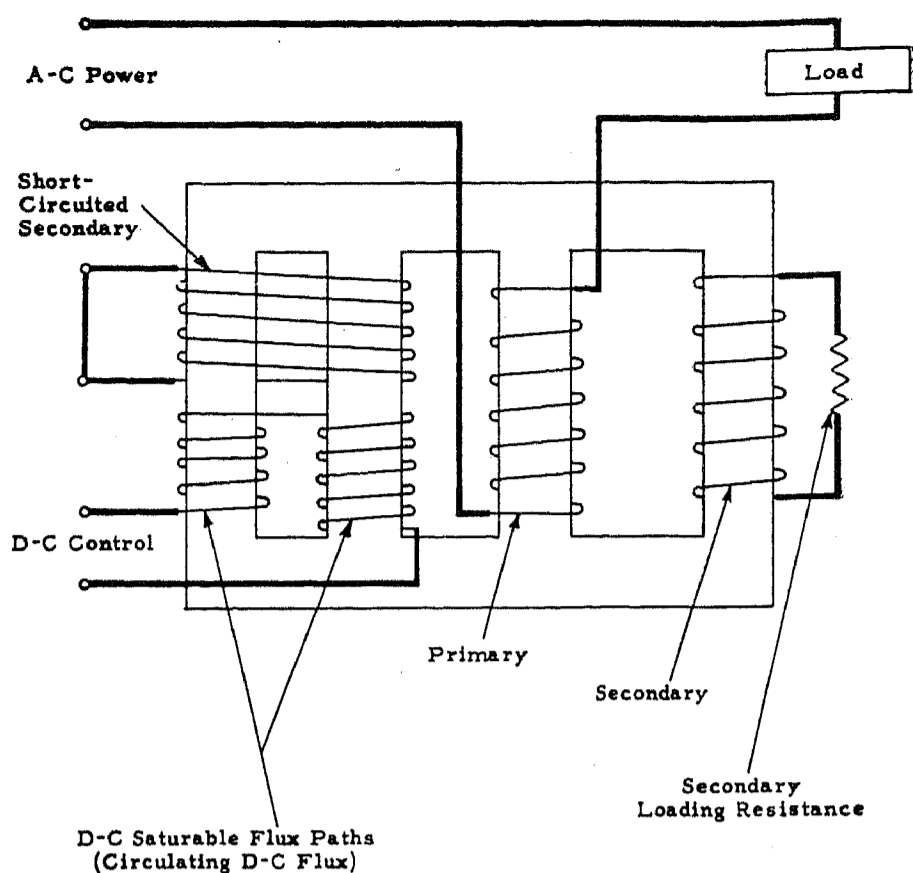


Figure 3. Variable mutual-inductance type d-c saturable transformer

Figure 2 is an example of a d-c controlled saturable transformer of the variable incremental-permeability type (class B', 1). This circuit is due to A. E. Bowen,<sup>2</sup> United States Patent 2,230,558 (1941). Here the primary and secondary windings are wound on the outside legs of a 3-legged core in such directions as to be mutually coupled and so as to pass no flux of power-supply fundamental frequency through the center leg of the core. Control is effected by imposing, via the saturating winding, a condition of d-c premagnetization. If a core is premagnetized by a d-c component of magnetomotive force, a given a-c magnetomotive force introduced by the primary winding will effect less change in flux than if the core were not premagnetized. Secondary voltage therefore may be controlled in a core by introducing a controlled amount

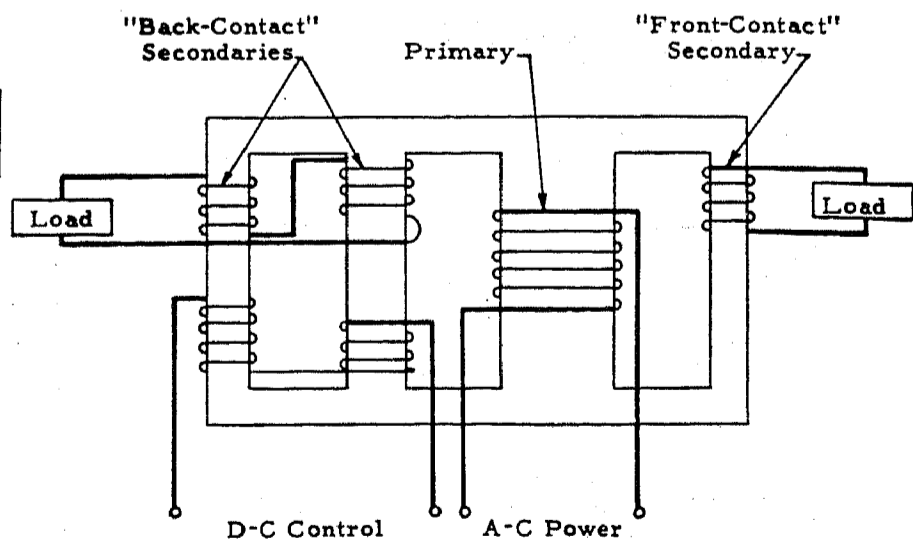
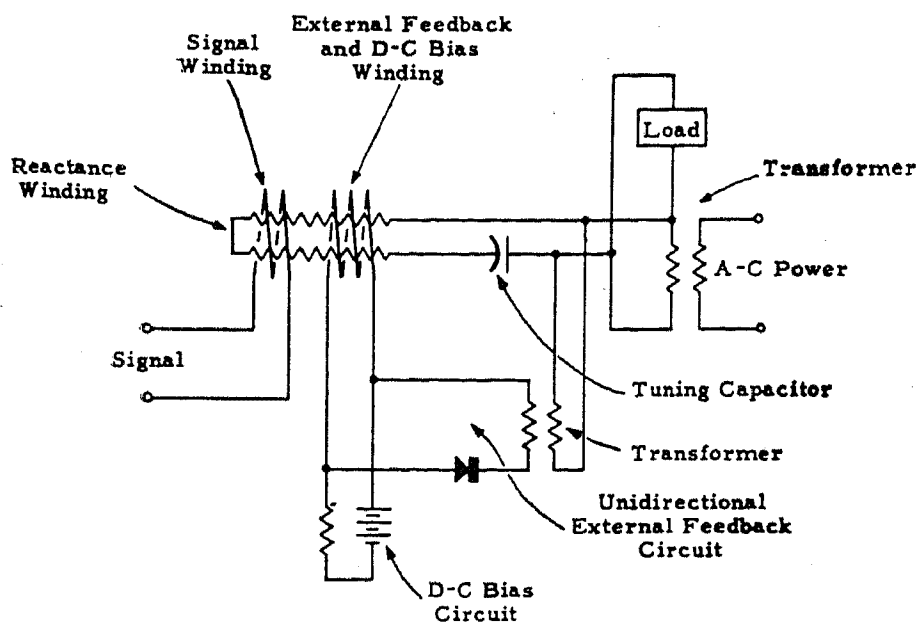


Figure 4. Variable mutual-inductance type d-c saturable transformer



of d-c premagnetization so as to effect a control of the a-c flux variation in the magnetic circuit linking the secondary winding with the primary winding. The amplitude of the a-c flux which mutually couples the primary and secondary is controlled by the degree of d-c premagnetization.

A d-c saturable-shunt type of variable-mutual-inductance transformer is shown in Figure 3. This circuit, due to E. R. Stoekle,<sup>3</sup> United States Patent 1,376,978 (1921), (class B", 1), shows a 4-legged transformer in which the flux that couples the secondary with the primary may be shunted, to a controlled degree, by a variable reluctance shunt. The mutual flux that links the secondary with the primary winding thus is decreased and controlled effectively. Control is effected in this typical circuit by means of saturating windings arranged to cause "circulating" d-c flux in the two saturable shunt legs. In this manner, the reluctance of the shunt legs to the passage of a-c flux is controlled effectively. In Figure 4, the transformer primary winding is inserted as a controlled reactance in series between

the load and a-c power source. The reactance of the primary winding in this circuit is controlled according to the combined reflected impedances from the two secondary windings with which it is immediately coupled, the reflected impedances being controlled according to the relative mutual coupling between the primary and secondary windings.

Another example of the d-c saturable shunt-type variable-mutual-inductance transformer is that shown in Figure 4. C. M. Hines<sup>4</sup> in United States Patent 2,215,820 (1940) described a saturable transformer (class B", 1) similar to that shown in Figure 3 but having another set of secondary windings placed on the d-c saturable legs as shown. These "back contact" secondary windings are arranged to be coupled effectively with the primary windings when the control legs are desaturated. On the other hand, if the d-c control legs are saturated, a minimum of primary flux will link the back contact secondary windings and a maximum of primary flux will link the "front contact" secondary winding. This circuit, with the primary winding excited by the a-c power

Figure 6 (below). Impedance-coupled external feedback means

Figure 8 (right). Internal-feedback magnetic amplifier with saturating winding control for a-c load

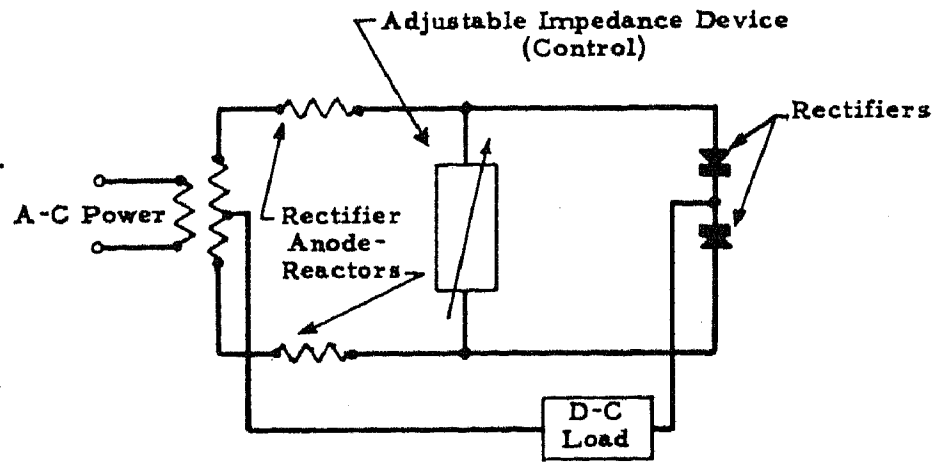
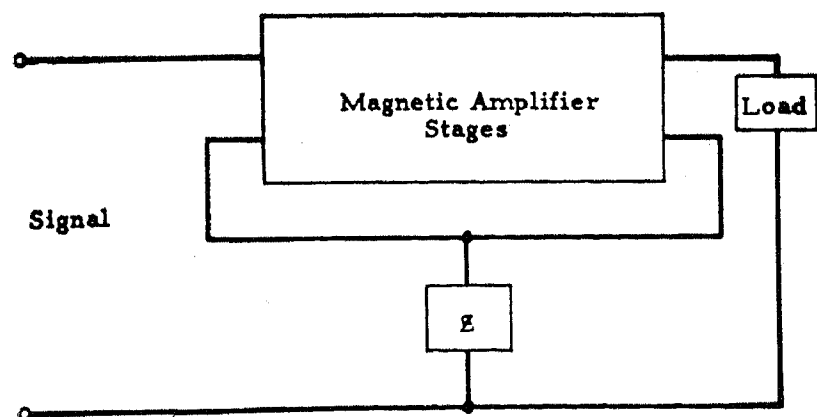


Figure 5 (right). Single-sided d-c saturable reactor with combined external feedback

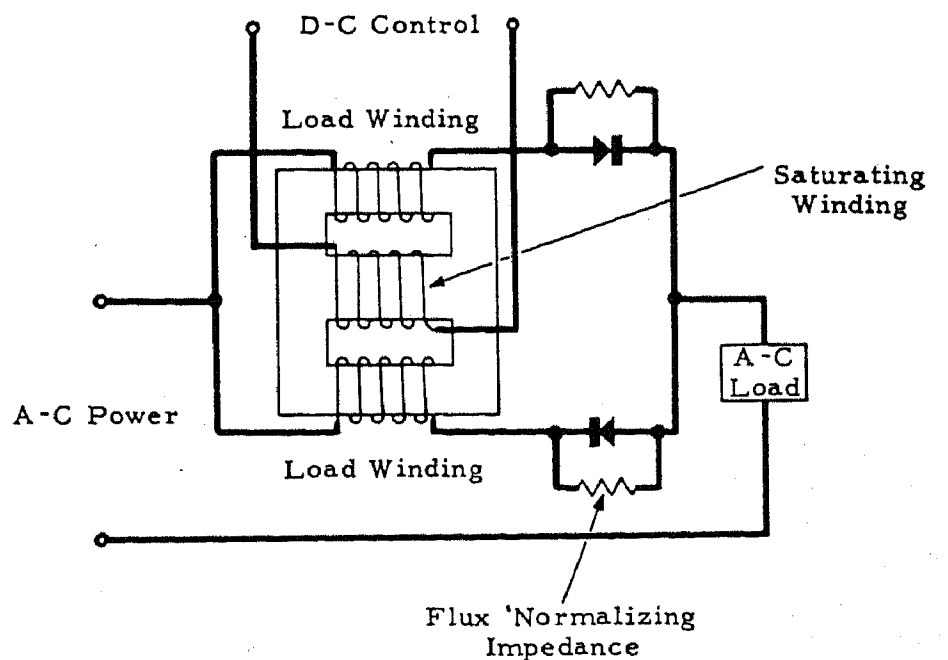
Figure 7 (above). Flux-normalizing method of control for internal-feedback magnetic amplifier

supply and with the secondary windings exciting separate loads, is, in effect, a single-pole double-throw a-c power switch.

Means are abundant in the magnetic-amplifier art by which either positive or negative feedback may be introduced to affect gain, stability, linearity, and speed of response of magnetic amplifiers. Feedback means may be divided into two principal classes: external feedback and internal feedback. These classes are described in the following paragraphs.

In external feedback circuits, a measure of output voltage and/or current is re-introduced as a feedback magnetomotive force via an external circuit which may terminate either in the control winding or in a special feedback winding. The amount of external feedback may be modified by adjusting either the amplitude or the phase of the feed-back signal with respect to the input or control signal.

One of the earliest examples of the application of amplitude-related external feedback to a d-c saturable-reactor magnetic amplifier is shown in Figure 5 (class A,1,a,c,f). Here E. F. W. Alexanderson<sup>5</sup>



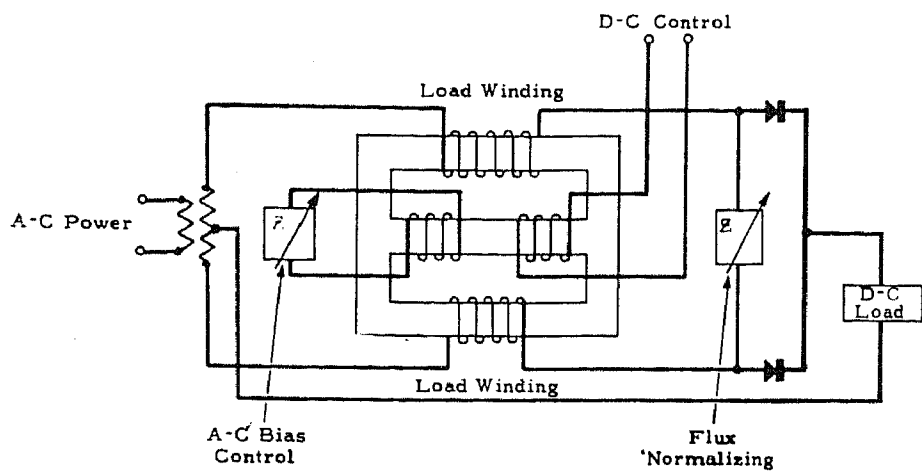


Figure 9. Internal-feedback magnetic amplifier with saturating winding control for d-c load

in United States Patent 1,328,797 (1920) showed a method for obtaining load-current control by placing a controlled reactance winding in parallel with the load and across the secondary terminals of an a-c supply transformer. A measure of magnetic-amplifier output potential (also load potential) is applied via a second transformer and rectifier circuit to an external feedback winding in such manner as to aid the effect of the control magnetomotive force. D-c bias is introduced via a parallel circuit using the same winding as used for external feedback, as shown.

An alternative means for introducing amplitude-related external feedback is shown in Figure 6 which illustrates, in block diagram form, an impedance-coupled external feedback means (class -, 1,a) described by A. S. FitzGerald<sup>6</sup> in United States Patent 2,464,639 (1949). A measure of load current creates a potential drop across the impedance  $Z$ . The impedance is so connected that the potential drop across it will aid (or oppose) the con-

trol signal applied at the magnetic-amplifier input. This action amounts to feedback.

A phase-related external feedback circuit is described by J. Slepian<sup>7</sup> in United States Patent 1,645,302 (1927), not illustrated here. In this patent a series-resonant inductance - capacitance phase shifter in the control circuit fixed the phase of the control signal with respect to the feed-back signal.

The so-called "internal feedback" means is the second important means for effecting feedback. In order to achieve internal feedback, rectifiers are inserted in series between the load windings and the load circuit. Since only unidirectional current impulses can flow in the load windings of such circuits, unidirectional pulsating flux components are created in the reactor cores in magnitude according to the magnitude of the load current. Control of such devices is effected in one of several ways to be discussed in the following paragraphs.

One of the earliest described means for effecting control in an internal feedback circuit is shown in Figure 7 (class A, 1,b,d). This method is due to F. G. Logan,<sup>8</sup> United States Patent 1,997,179 (1935). A "normalizing shunt" is placed in shunt with the rectifier circuit so that a controlled amount of reverse magnetomotive force may be applied to the reactor core during its quiescent half-cycle. By means of this shunt circuit, a current is caused to flow at such times as the reactor otherwise would be inactive, the direction of this current being opposite to that in which the current will flow when the reactor again becomes active in carrying load current. The flux and energy conditions thus tend to establish, to a degree determined by the adjustment of the normalizing impedance, the conditions which would have existed if the reactor had not been subjected to an inactive condition. According to the value of the normalizing impedance, therefore, a controlled amount of energy and current is delivered by the load circuit. The changes in normalizing-impedance current required to effect control are claimed to be very slight compared to the changes in load current which may be obtained.

Another and more widely used means of control for internal-feedback magnetic amplifiers is shown in Figures 8 and 9. This means of magnetic-amplifier control employs a d-c saturating winding to pre-saturate the core by an amount adjusted by the control current. The control winding links the magnetic core arrangement in such manner that the d-c control mag-

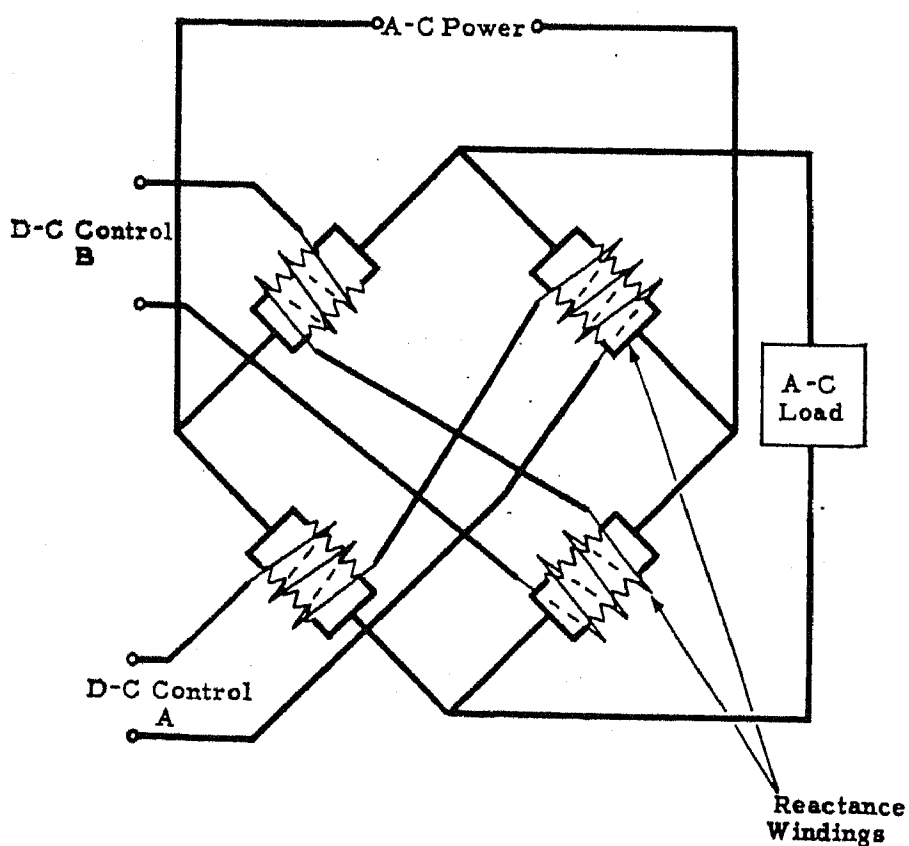


Figure 10. Wheatstone-bridge type magnetic amplifier

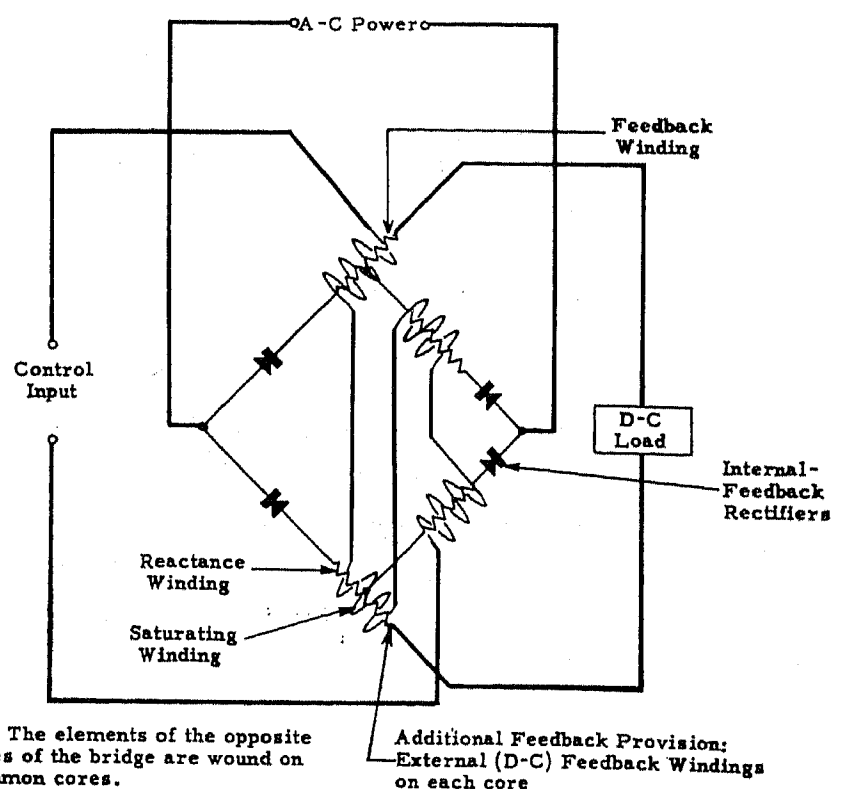


Figure 11. Wheatstone-bridge type magnetic amplifier with internal and external feedback

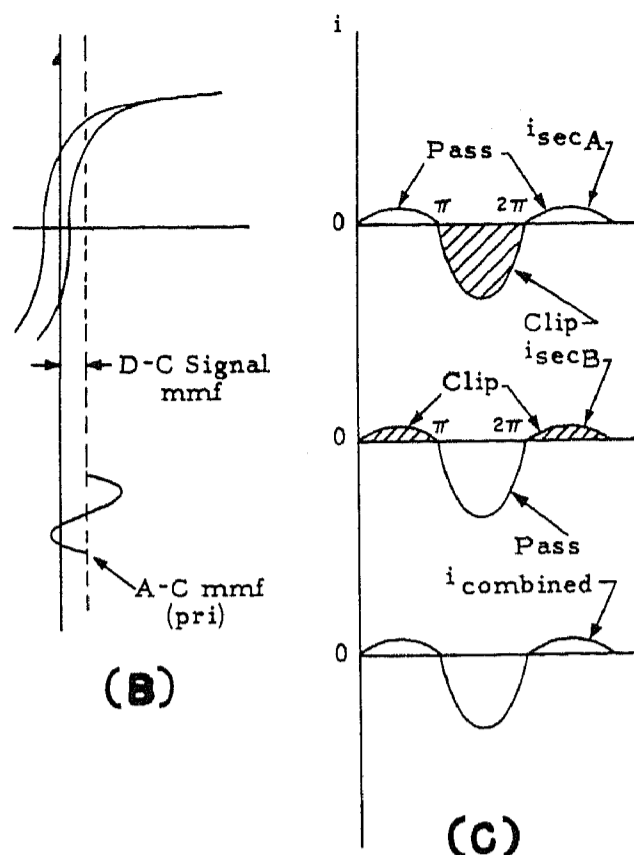
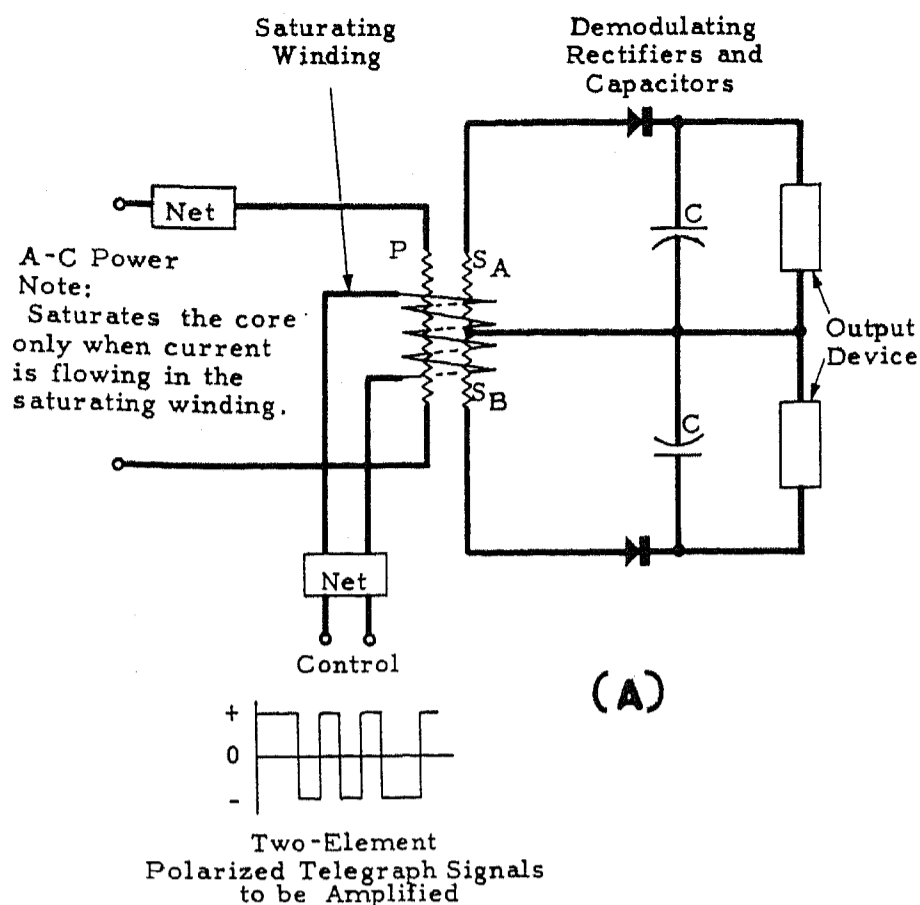


Figure 12. Push-pull magnetic amplifier with internal feedback and with frequency-discriminating means

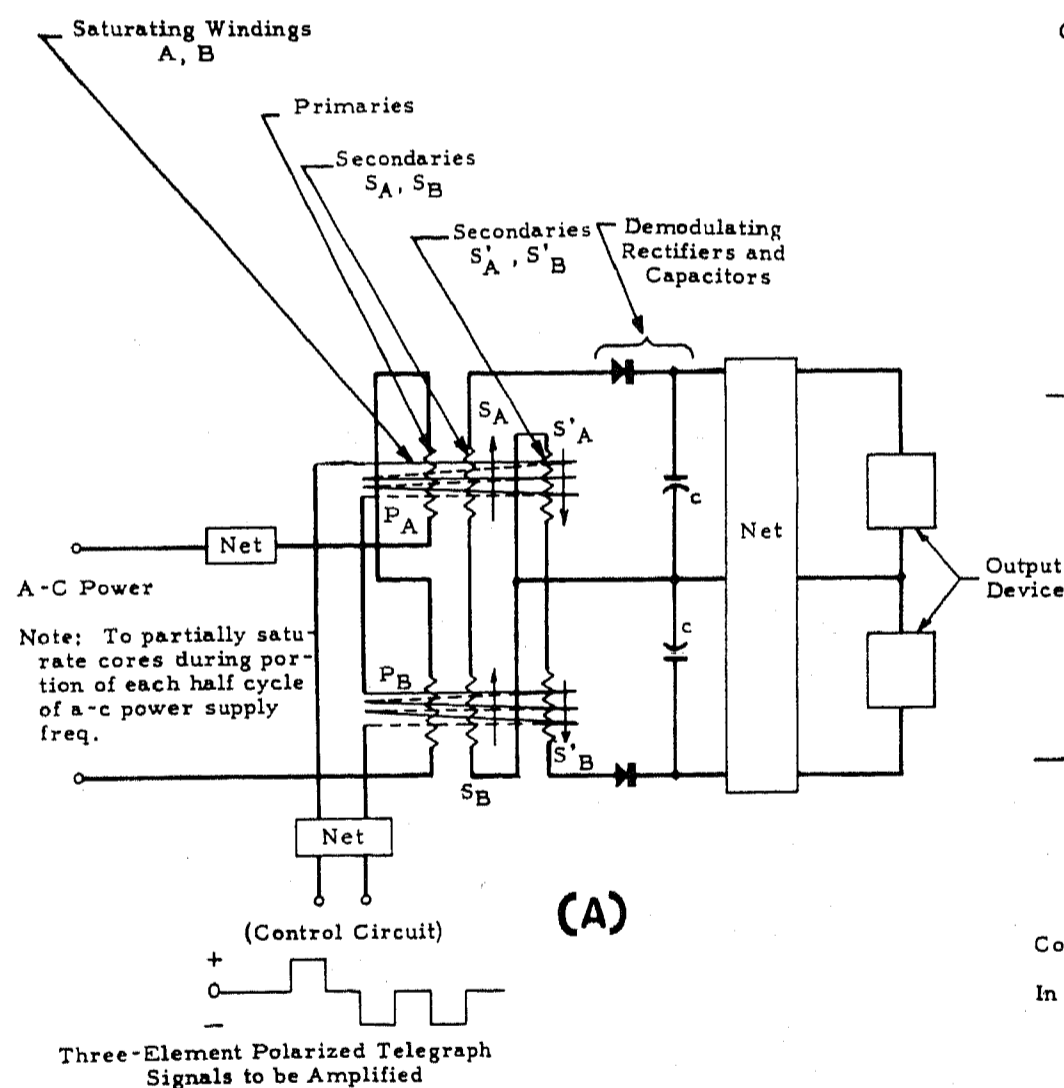
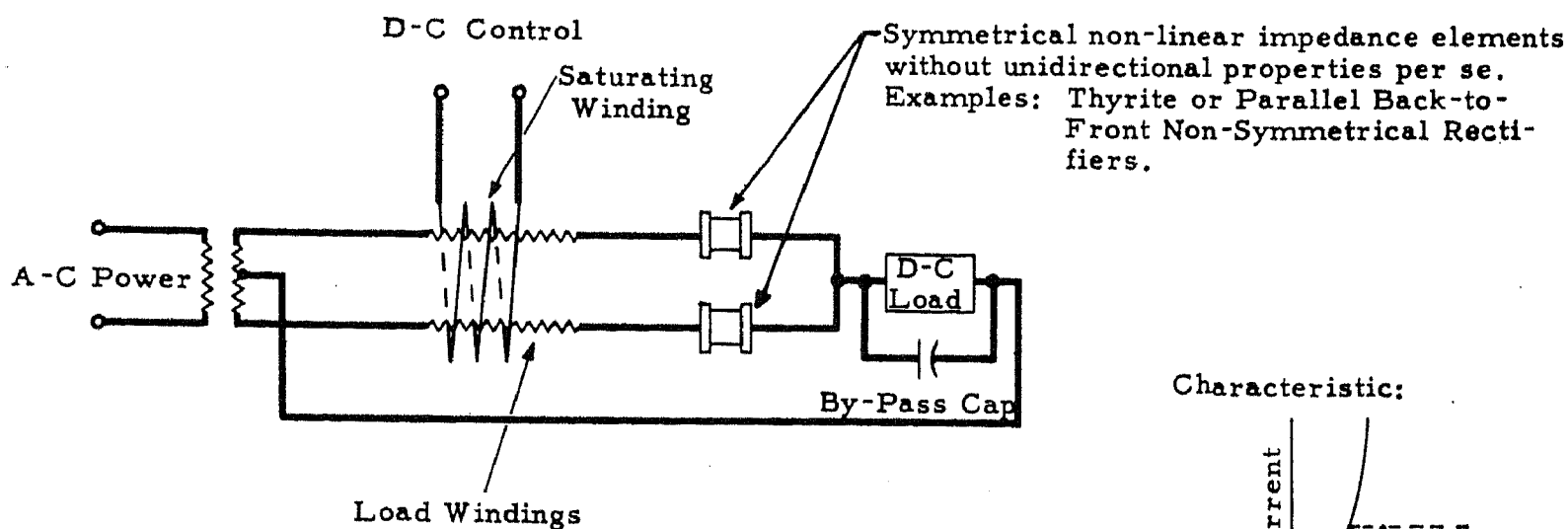


Figure 13. "Second-harmonic type" push-pull magnetic amplifier with internal feedback and with frequency-discriminating means

netomotive force either will be aided or opposed by the unidirectional internal-feedback magnetomotive force set up by the load windings. Figure 8 shows an internal-feedback magnetic amplifier (class A,1,b,d) for the control of an a-c load. This circuit is due to F. G. Logan,<sup>9</sup> United

States Patent 2,126,790 (1938). Figure 9 illustrates a similar d-c signal-winding-controlled internal-feedback magnetic amplifier (class A,1,b,d) for the control of a d-c load circuit. This circuit was described by S. M. Hanely,<sup>10</sup> United States Patent 2,144,290 (1939).

Magnetic amplifiers may be arranged in the form of a Wheatstone bridge. Such circuits are advantageous in that they may be set up readily to have polarized characteristics, that is, they may be made to deliver positive output corresponding to a positive control signal, zero output



Note: When the core is under the influence of control mmf, A-C Power Supply is sufficient to saturate appreciably the core once each cycle.

Figure 14. Unbalanced harmonic device employing

1. Symmetrical current-distorting impedances
2. Asymmetrical current-distorting impedances with polarizing means (d-c saturable reactors)

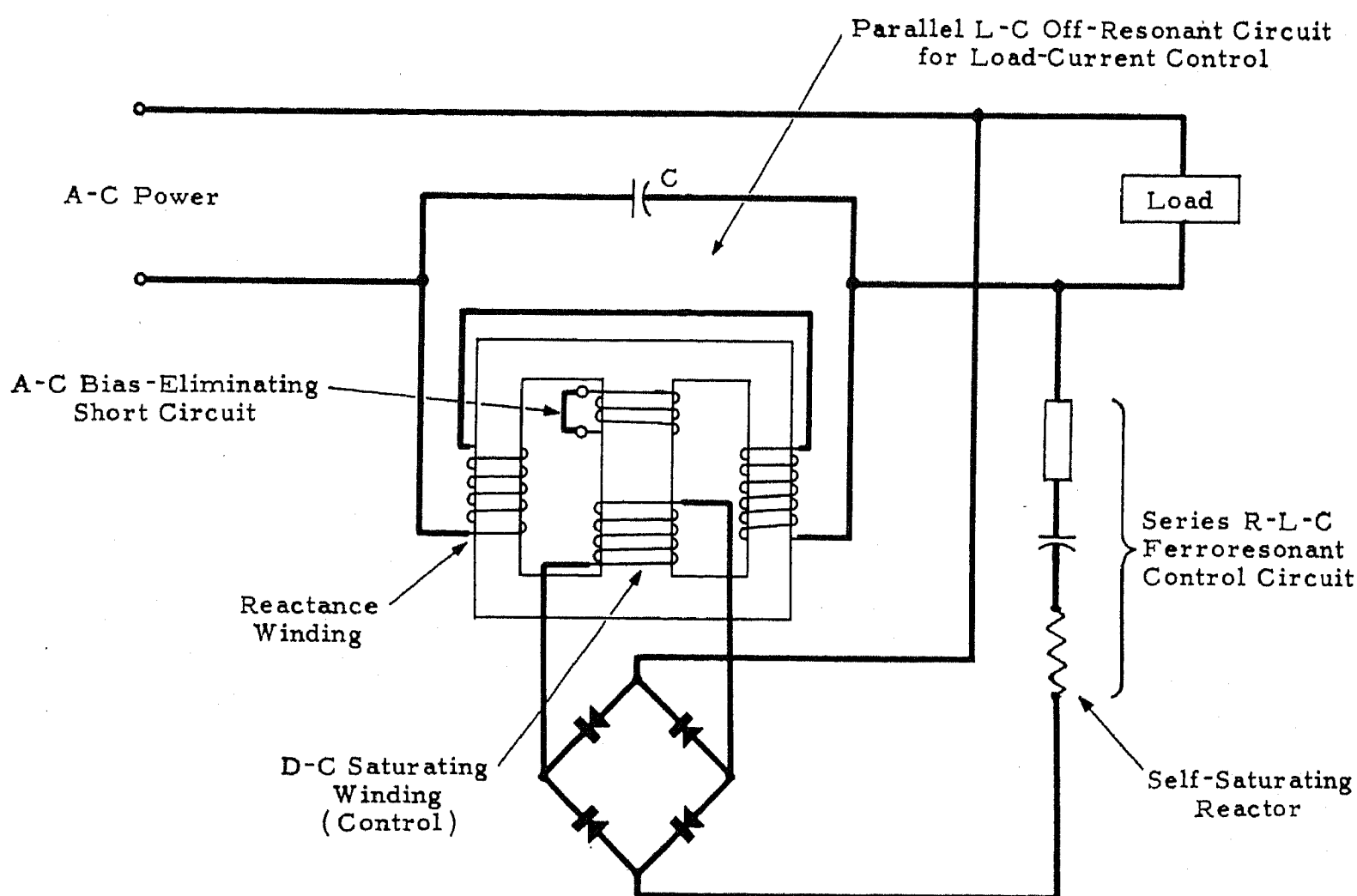
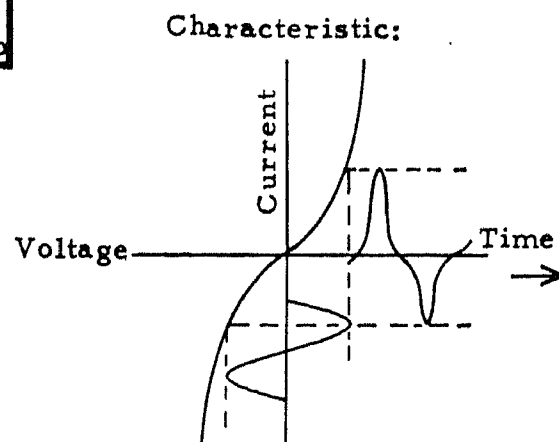


Figure 15. Regulating apparatus with

1. Inductance-capacitance off-resonant variable-impedance circuit
2. Resistance-inductance-capacitance ferroresonant control circuit

corresponding to zero control signal, or negative output current corresponding to a negative control signal. These bridges have the advantage that the control, output, and power-supply circuits are practically isolated one from the other.

Many Wheatstone-bridge circuits have been described in which one or more arms of the bridge comprise the reactance windings of magnetic amplifiers. A specific example is the class A,3 circuit shown in Figure 10, this circuit being due to L. W. Thompson,<sup>11</sup> United States

Patent 1,844,704 (1932). All four arms of the bridge are the reactance windings of d-c magnetomotive-force-controlled saturable reactors. Internal and external feedback means are shown in Figure 11 as applied to the Wheatstone-bridge type magnetic amplifier due to Uno Lamm,<sup>12</sup> United States Patent 2,403,891 (1946) (class A,3,a,b).

A class B',2,b,f,g device is disclosed by E. M. Boardman<sup>13</sup> in United States Patent 2,108,642 (1938). This device, schematically shown in Figure 12(A), is for

repeating and amplifying 2-element polarized signals such as those used in telegraphy. The magnetic amplifier comprises a single core in the form of a d-c saturable transformer having a control or saturating winding, a primary winding, and two secondary windings connected in a circuit which is the equivalent of a push-pull system. The amount of a-c power magnetomotive force introduced via the primary winding normally is not sufficient to saturate the core unless control magnetomotive force also exists.

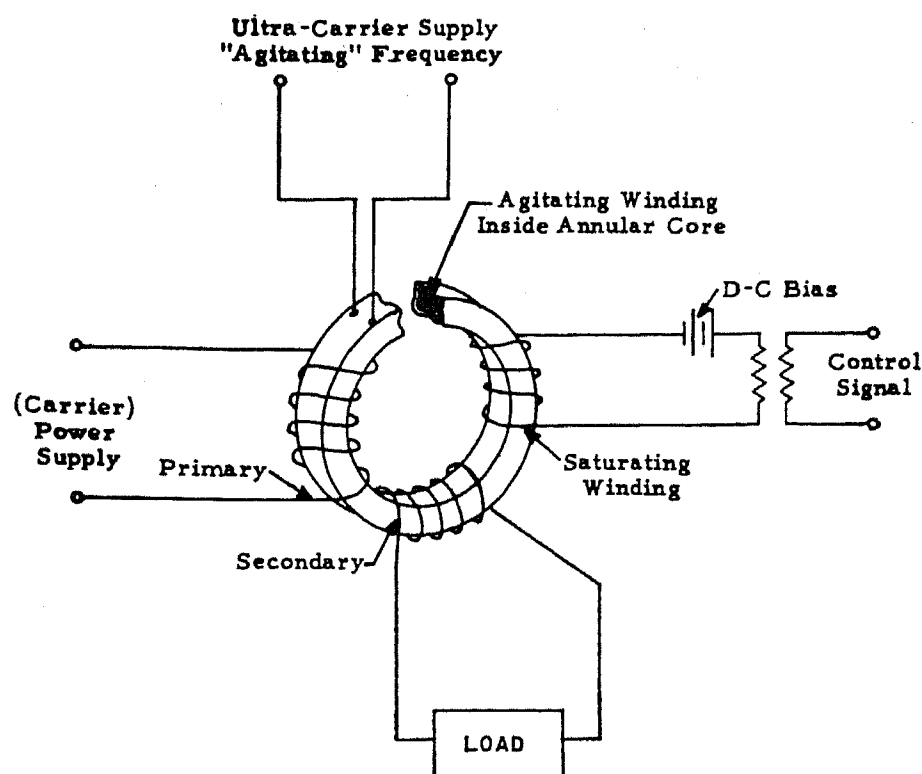


Figure 16. Magnetic amplifier with ultracarryer frequency "agitating" means for hysteresis-effect reduction

If the control signal is zero, equal and opposite potentials of the same frequency as the power-supply frequency are induced in the secondary windings. Equal and opposite currents thus are caused to flow through the rectifiers and through the two halves of the output device; the net output signal is consequently zero.

Assume that a positive control current is made to flow in the saturating winding. The wave shapes of the voltages induced in the secondary windings will now assume the forms shown in Figure 12(C) because of the displaced excitation conditions illustrated in Figure 12(B). The rectifier associated with secondary  $S_A$  will pass positive half-cycles of current, and the rectifier associated with secondary  $S_B$  will pass negative half-cycles. Because these rectifiers exhibit a considerably reduced impedance when the applied forward voltage is increased, a notably greater current flows through the rectifier which is conducting than through the rectifier which is not conducting. Due both to the distortion in secondary voltage wave form and to the nonlinear resistance characteristic of the rectifiers, therefore, a considerable unbalance of voltage drop is made to occur between the two halves of the output device.

The unidirectional currents flowing through the output devices also flow in secondary windings  $S_A$  and  $S_B$  in unbalanced magnitudes such that a net unidirectional magnetomotive force exists in a direction aiding the original control magnetomotive force. This amounts to positive internal feedback which increases the amplifier output.

Negative control-current signals have similar, though opposite, effects.

A magnetic flip-flop based on the principle of the circuit of Figure 12, but with external feedback added is described by E. T. Burton<sup>14</sup> in United States Patent 2,147,688 (1939).

Figure 13 is the schematic of a unique magnetic amplifier claimed by E. M. Boardman<sup>13</sup> in United States Patent 2,108,642 (1938) and belonging to class  $B'$ , 2, b, f, g. Before demodulation the amplified signal is an amplitude-modulated wave form whose carrier frequency is double that of the a-c power-supply frequency. This type has been called a "second harmonic device" by some writers.

The magnetic amplifier of Figure 13 is claimed to be particularly adaptable for repeating and amplifying 3-element signal impulses, that is, positive, zero, and negative impulses. Two cores in the form of d-c saturable transformers are employed. Each core has a control or saturating winding, a primary winding, and two secondary windings connected in the

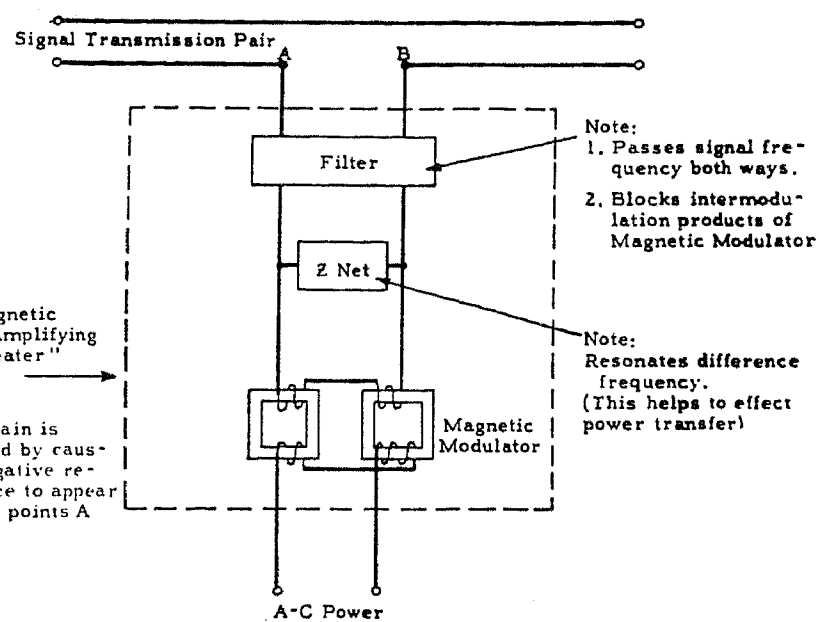
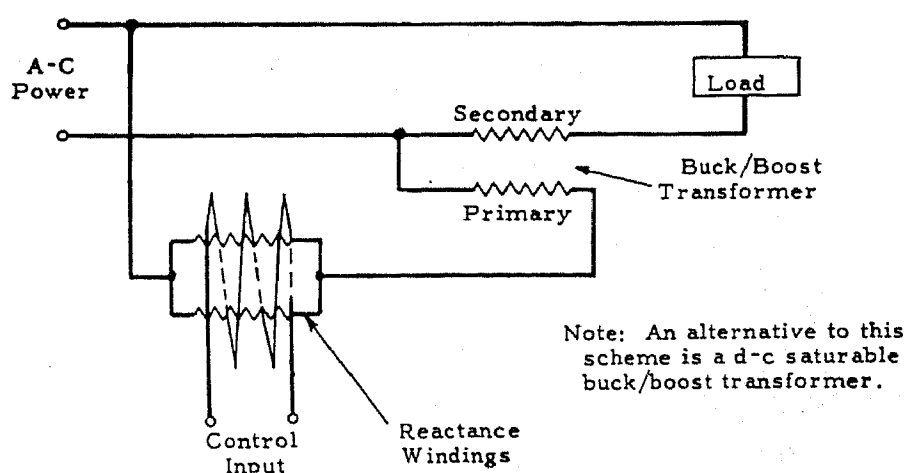


Figure 17. Series negative-resistance type magnetic amplifier

equivalent of a push-pull system. A-c power is introduced via the two primary windings, and the power-supply frequency is considerably higher than any component of the input signal frequency. An essential feature of the primary excitation of this circuit is that it must be sufficient to saturate the cores partially during a portion of each half-cycle of the power-supply frequency. The saturating windings are connected so that, with zero control signal, equal and opposing electromotive forces of power-supply frequency are induced in them. Decoupling of the control circuit from the a-c power supply is provided in this manner.

Parameters are chosen so that, with zero input signal, potentials of equal and opposite wave forms containing a minimum of even harmonic distortion are induced in the seriesed secondaries. The resulting potential across the seriesed secondaries is therefore zero under zero signal conditions, and no current flows in either of the output circuits.

When a positive current impulse, as shown in Figure 13(B), is applied to the saturating windings, however, the net operating points on the magnetic characteristics of the two cores are shifted so that potentials induced in the secondaries



Note: An alternative to this scheme is a d-c saturable buck/boost transformer.

Figure 18. Buck/boost transformer controlled by d-c saturable reactor

of one core are no longer exactly opposed to the potentials induced in the secondaries on the other core. Instead, a wave form with strong second-harmonic content is induced in both secondary windings on each core, and the series combination of one secondary on one core and one secondary on the other core yields a net electromotive force having essentially second-harmonic wave form. The displaced excitation conditions are illustrated in Figure 13(B), and the individual and net effects are illustrated in Figure 13(C). The second-harmonic wave form of voltage is applied to the demodulators in the respective halves of the output circuit. One rectifier is connected in the output circuit comprising windings  $S_A$  and  $S_B$  in such a manner that it will conduct under the influence of high-potential positive peaks induced in this circuit. Another rectifier is oppositely connected in the output circuit comprising windings  $S_A'$  and  $S_B'$  in a direction so that it will conduct under the influence of the negative peaks, but not the positive peaks. The rectifier elements have nonlinear characteristics (copper-oxide rectifiers, for example) in which the forward impedance decreases rapidly as forward potential increases. The forward current is thus much greater through the conducting rectifier than is the reverse current through the nonconducting rectifier. Therefore, because of the combined characteristics of the circuit, the upper half of the output device is predominantly energized.

The unidirectional demodulated currents flowing through the output devices also flow through the secondary windings in the direction shown by the arrows in Figure 13(A). The unidirectional current flow in the secondaries produces magnetomotive forces which in  $S_A$  and  $S_B$  aid the magnetomotive force due to the positive control current applied to the saturating windings, but in  $S_A'$  and  $S_B'$  the unidirectional current produces a magnetomotive force which opposes the control magnetomotive force. Since the current in windings  $S_A$  and  $S_B$  greatly exceeds that in windings  $S_A'$  and  $S_B'$ , however, the net effect is that the control magnetomotive force is aided by the feedback magnetomotive force which results from the flow of demodulated output current in the secondary windings. This action amounts to positive internal feedback and operates to increase the output of the amplifier.

Negative current impulses applied to the control circuit have similar though opposite effects.

A group of devices similar to that shown in Figure 13 are the subjects of E. T. Burton's United States Patents 2,147,688 (1939)<sup>14</sup> and 2,164,383 (1939)<sup>15</sup> which include magnetic flip-flops. These devices are not illustrated in this report.

An important feature of the devices shown in Figures 12 and 13 is the inclusion of the frequency-discriminating networks labelled "net" in the control, power supply, and output circuits. The purpose of the network shown in the a-c power-supply circuit is presumably to resonate the

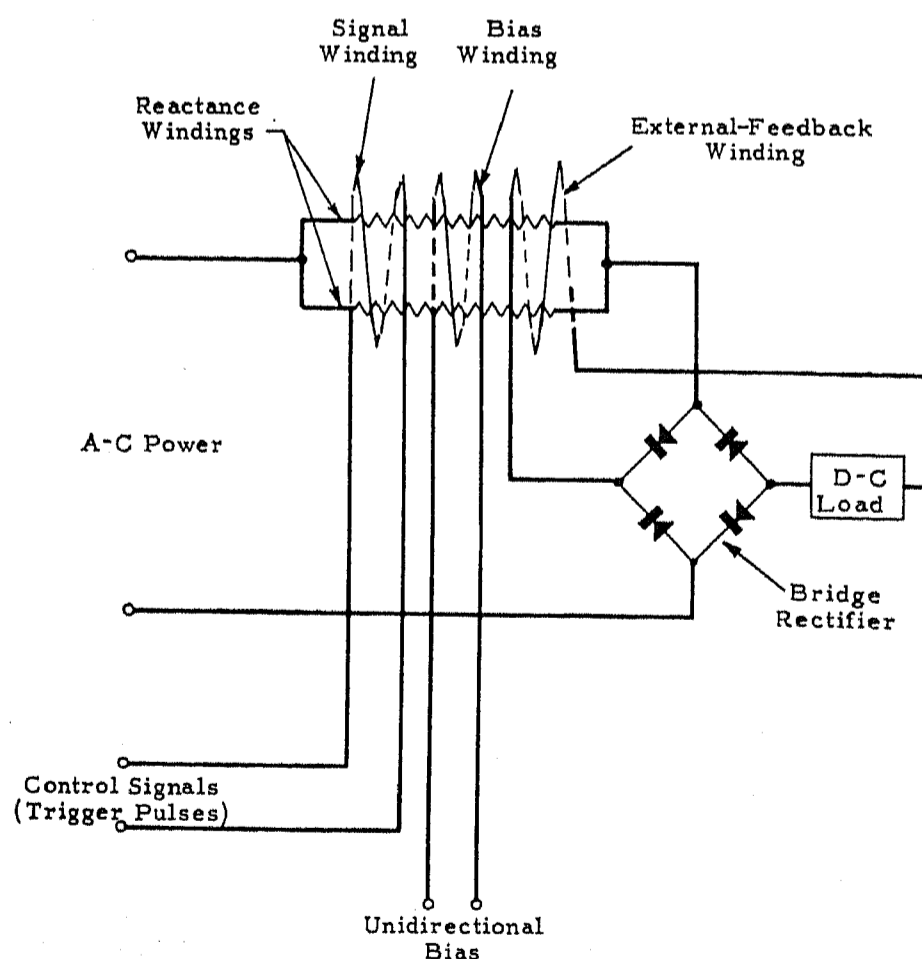
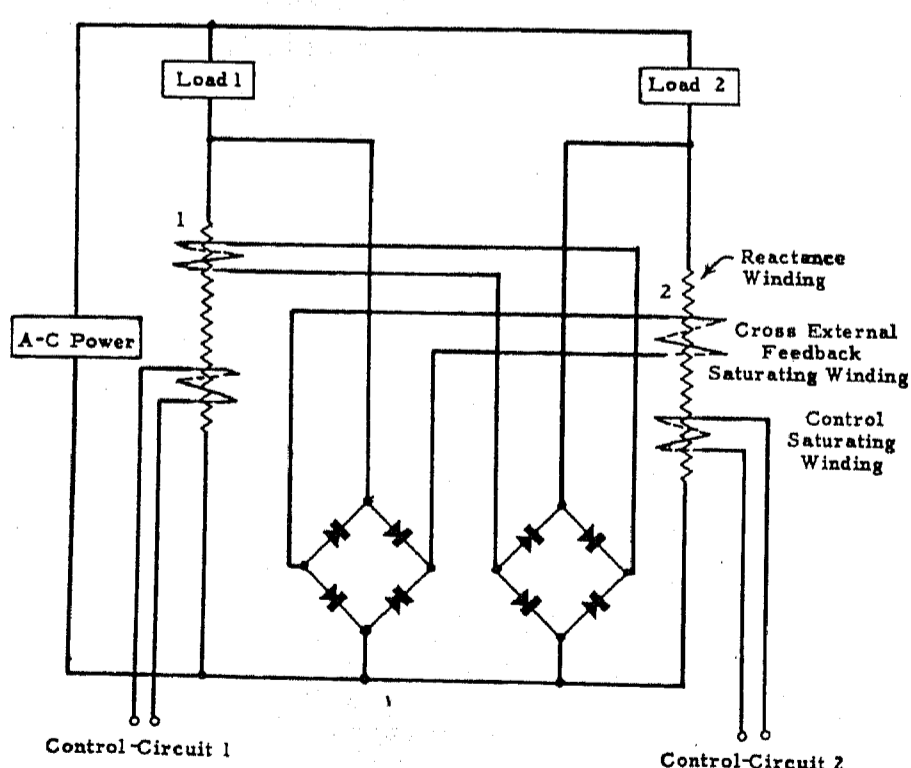
primary circuit to the frequency of the a-c power supply (a series capacitor for the same purpose is included in the reactor circuit of Figure 6). The purpose of the network in the control circuit is to prevent the power-supply frequency harmonic voltages induced in the saturating windings from affecting the control circuit, and to prevent changes in the impedance of the control line from affecting the operation of the amplifier. The network in the output circuit, Figure 13, is to improve the wave forms of the amplified signals which are transmitted to the output device.

A class  $A,1,g$  magnetic amplifier, which is possibly unique in the art, is shown in Figure 14. Described by H. D. Middel<sup>16</sup> in United States Patent 2,388,070 (1945), this device comprises a d-c saturable reactor in which the saturating flux produced by the control winding causes a second harmonic electromotive force of power-supply frequency to be induced in the load windings. Under the influence of control saturation, an asymmetrical voltage is caused to act across a nonlinear resistor seriesed with the load winding of each stage. This departure from symmetry causes the nonlinear resistor to act as a rectifier which introduces a d-c component in the load circuit. A minute change of d-c signal applied to the saturating winding of the reactor will thus cause a much larger change in d-c load current.

The symmetrical nonlinear impedance elements in this device have no unidirectional

Figure 19 (below). Magnetic flip-flop, unbalanced d-c saturable-reactor type with cross-external feedback

Figure 20 (right). Magnetic amplifier (flip-flop), single-sided d-c saturable reactor type with external feed-back winding



tional property or rectifying tendency per se. For example, they may be either Thyrite or parallel back-to-back rectifiers such as copper-oxide rectifiers. The basic elements of the amplifier are thus a symmetrical current-distorting impedance, an asymmetrical current-distorting impedance with polarizing means, an a-c power supply, and a d-c load circuit. It may be that this device should be considered as a general case within which the conventional internal-feedback magnetic amplifier, as shown in Figure 10, is a special case. The reader is referred to the patent for a detailed description of this device.

Important off-resonant and ferroresonant means for increasing the gain of magnetic amplifiers have been described from an early date. Figure 15 shows an example of the use of such sensitivity increasing means in a (class *A,1,a,d,e*) voltage-regulating device due to L.R. Runaldue,<sup>17</sup> United States Patent 2,278,151 (1942). A parallel inductance-capacitance off-resonant circuit is used for load-current control. The range of impedance obtainable across the terminals of the parallel inductance-capacitance circuit as it approaches resonance is considerably greater than the range which might be obtained by use of the saturable reactor alone. The impedance of this network is changed by varying the inductance of the d-c saturable reactor. If the inductive reactance of the reactor approaches the capacitive reactance of the paralleled capacitor, the network will approach a resonant condition and the impedance will approach a very high value. If the inductive reactance of the reactor is either increased or decreased from the resonant value, the impedance of the network will decrease.

The sensitivity of the control circuit in Figure 15 may be increased by the inclusion of a ferroresonant combination comprising a series-connected capacitor, resistor, and self-saturating inductance. By use of the ferroresonant combination, the current in the control circuit under the influence of a varying voltage excitation may be made to increase at a much greater rate than the excitation voltage. In other words, the ferroresonant combination amplifies the current or voltage variations occurring in the circuit from which it is excited.

Figure 16 illustrates an unusual and little-exploited means for increasing the gain of a magnetic amplifier by altering the hysteresis effects of the core material. In this class *B',1,c,h*, device, due to R.V. L. Hartley,<sup>18</sup> United States Patent 1,287,982 (1918), the inventor describes means of altering the hysteresis effects in the

core material by the application of a special agitating field. The core is subjected to a weak high-frequency cross-magnetization at a frequency several times that of the a-c power supply (carrier). This patent appears to be based first on the principle that a distribution of hysteresis loss can be made between two frequencies (the agitating frequency and the carrier frequency), and second on the corollary principle that at low flux densities the dynamic incremental permeability of the material at the low frequency (carrier) is considerably increased by the presence of the higher frequency excitation. For a pertinent theoretical treatise on this phenomenon, see the article by R. M. Kalb and W. R. Bennett.<sup>19</sup> See also G.W. Elmen and C. F. Ort,<sup>20</sup> United States Patent 1,544,381, 30 June, 1925, which applies the phenomenon.

A significant and novel type of magnetic amplifier is the negative resistance type shown in Figure 17 and designated as class *C,1,f*. This device was described by E. Peterson<sup>21</sup> in United States Patent 1,884,844 (1932). The operation of this circuit depends upon the intermodulation of power-supply and signal frequencies in such manner that energy is transferred from the power source to the signal circuit. The energy transfer results when a difference side band of the two frequencies (for example, twice power frequency minus signal frequency) is permitted to flow in the circuit. See K. Heegner,<sup>22</sup> United States Patent 1,656,195 (1928). Other intermodulation products must not be allowed to flow and thereby to consume power. An impedance network allows signals of the difference side-band frequencies to flow internally in the device. Filters suppress the flow of signals of the other generated frequencies except the signal frequencies. A general limitation of the device is that the bandwidth of the signal must be less than the power-supply frequency; for wider bandwidths it appears that the suppression of unwanted frequencies will be increasingly difficult.

Classification 4 of the chart designates "special magnetic amplifier circuits." An example of a circuit of this class is shown in Figure 18 due to H.E. Young,<sup>23</sup> United States Patent 2,154,020 (1939). This is a class *A,1,4*, circuit comprising a d-c controlled saturable reactor which in turn controls a buck/boost transformer the secondary winding of which is inserted in series with the load. The characteristics of this circuit are such that zero load current may be obtained for zero control signal provided that the voltage induced in the secondary of the transformer is equal and opposite to the voltage of the

a-c power source under zero-signal conditions.

Another special circuit is that shown in Figure 19 due to M. L. Wood,<sup>24</sup> United States Patent 2,524,154 (1950). This is a magnetic-amplifier counterpart of the Eccles-Jordan trigger pair in which flip-flop action is obtained by using part of the output signal of one amplifier as the input signal to the other amplifier and vice versa. In this class *A,4,a* circuit, the reactance winding of saturable reactor 1 controls (via its associated rectifier) the d-c magnetomotive force applied via the saturating winding to core 2. Correspondingly, the reactance winding of saturable reactor 2 controls the d-c magnetomotive force applied via the saturating winding to core number 1. This flip-flop may be said to be a "cross external-feedback type" of magnetic amplifier. The state of this flip-flop may be changed by applying unidirectional magnetomotive forces by means such as the control windings shown.

Another type of magnetic flip-flop is shown in Figure 20. This is a single-sided magnetic amplifier similar to that shown in Figure 5, but with a sufficient amount of external feedback to cause instability. This class *A,1,a,c* device is described by FitzGerald<sup>25</sup> in United States Patent 2,027,312 (1936). It has been suggested that the signal-sided flip-flop circuit of Figure 20 may be used to advantage in that it does not require equal loads as does the balanced Eccles-Jordan type flip-flop of Figure 19.

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